Nonlinear Stability of Rarefaction Waves for Boltzmann Equation

Tai-Ping Liu*
Institute of Mathematics, Academia Sinica, Taipei
and
Department of Mathematics, Stanford University

Tong Yang†
Department of Mathematics, City University of Hong Kong

Shih-Hsien Yu‡
Department of Mathematics, City University of Hong Kong

Hui-Jiang Zhao§
Wuhan Institute of Physics and Mathematics
The Chinese Academy of Sciences, Wuhan 430071, China
and
School of Political Science and Economics
Waseda University, Tokyo 169-8050, Japan

Abstract

It is well-known that Boltzmann equation is closely related to the systems of fluid dynamics which have rich nonlinear wave phenomena. In this paper, we study the nonlinear stability of a nonlinear wave pattern consisting two rarefaction waves for the Boltzmann equation. The analysis combines the analytical techniques used in the study of conservation laws with the micro-macro decomposition for the Boltzmann equation introduced in [18] and [19] through energy method.

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*Email address: liu@math.stanford.edu. Research supported by Institute of Mathematics, Academia Sinica and NSF Grant DMS-9803323.

†Email address: matyang@math.cityu.edu.hk. Research supported by the Strategic Research Grant of City University of Hong Kong # 7001439.

‡Email address: mashyu@math.cityu.edu.hk. Research supported by the Competitive Earmarked Research Grant of Hong Kong #9040645.

§Email address: hhjjzhao@hotmail.com. Research supported by the JSPS Research Fellowship for Foreign Researchers, the National Natural Science Foundation of China under contracts 10001036 and 10041003 respectively, and the grant from the Chinese Academy of Sciences entitled "Yin Jin Guo Wai Jie Chu Ren Cai Ji Jin".

1 Introduction

The one dimensional Boltzmann equation takes the form of

$$f_t + \xi_1 f_x = Q(f,f),\ (f,t,x,\xi) \in \mathbf{R}\times\mathbf{R}^+\times\mathbf{R}\times\mathbf{R}^3, \tag{1.1}$$

where $f(t,x,\xi)$ represents the distributional density of particles at time-space (t,x) with velocity ξ and $Q(f,f)$ is a bilinear collision operator, cf. [3]. In the following, we consider the hard sphere model, for which $Q(f,g)$ is:

$$Q(f,g)(\xi) \equiv \frac{1}{2}\int_{\mathbf{R}^3}\int_{\mathbf{S}^2_+} \Big(f(\xi')g(\xi'_*)+f(\xi'_*)g(\xi')-f(\xi)g(\xi_*)-f(\xi_*)g(\xi)\Big)\,|(\xi-\xi_*)\cdot\Omega|\ d\xi_* d\Omega.$$

Here $\mathbf{S}^2_+ = \{\Omega\in\mathbf{S}^2:\ (\xi-\xi_*)\cdot\Omega \ge 0\}$ and

$$\begin{cases} \xi' = \xi - [(\xi-\xi_*)\cdot\Omega]\,\Omega, \\ \xi'_* = \xi_* + [(\xi-\xi_*)\cdot\Omega]\,\Omega. \end{cases}$$

It is well-known that the Boltzmann equation is closely related to the systems of fluid dynamics, i.e., Euler equations and Navier-Stokes equations, which have two types of nonlinear waves, shocks and rarefaction waves. The existence and nonlinear stability of shock waves have been studied extensively, cf. [4, 19] and references therein. In this paper, we will study the nonlinear stability of the other type of nonlinear waves, i.e., rarefaction waves. Our analysis is based on the micro-macro decomposition of the Boltzmann equation introduced in [18] and [19] which rewrites the Boltzmann equation into a system of the type of fluid dynamics. And then the approach to the nonlinear stability of rarefaction wave for fluid dynamics can be applied here so that both the dissipations of fluid and non-fluid components are used.

Precisely, we decompose the solution of the Boltzmann equation $f(t,x,\xi)$ into the macroscopic (fluid) component, i.e., the local Mawellian $\mathbf{M} = \mathbf{M}(t,x,\xi) = \mathbf{M}_{[\rho,u,\theta]}(\xi)$; and the microscopic (non-fluid) component, i.e., $\mathbf{G} = \mathbf{G}(t,x,\xi)$ as follows:

$$f(t,x,\xi) = \mathbf{M}(t,x,\xi) + \mathbf{G}(t,x,\xi).$$

The local Maxwellian $\mathbf{M}$ is defined by the five conserved quantities, that is, the mass density $\rho(t,x)$, momentum $m(t,x)=\rho(t,x)u(t,x)$, and energy $\mathbf{E}(t,x)+\frac{1}{2}|u(t,x)|^2$ defined by:

$$\begin{cases} \rho(t,x) \equiv \int_{\mathbf{R}^3} f(t,x,\xi)d\xi, \\ m^i(t,x) \equiv \int_{\mathbf{R}^3}\psi_i(\xi)f(t,x,\xi)d\xi \text{ for } i=1,2,3, \\ \left[\rho\left(\mathbf{E}+\frac{1}{2}|u|^2\right)\right](t,x) \equiv \int_{\mathbf{R}^3}\psi_4(\xi)f(t,x,\xi)d\xi, \end{cases} \tag{1.2}$$

$$\mathbf{M} \equiv \mathbf{M}_{[\rho,u,\theta]}(t,x,\xi) \equiv \frac{\rho(t,x)}{\sqrt{(2\pi R\theta(t,x))^3}}\exp\left(-\frac{|\xi-u(t,x)|^2}{2R\theta(t,x)}\right). \tag{1.3}$$

Here $\theta(t,x)$ is the temperature which is related to the internal energy $\mathbf{E}$ by $\mathbf{E}=\frac{3}{2}R\theta$ with R being the gas constant, and $u(t,x)$ is the fluid velocity. And $\psi_\alpha(\xi)$ are the five collision invariants, cf. [3]:

$$\begin{cases} \psi_0(\xi) \equiv 1, \\ \psi_i(\xi) \equiv \xi^i \quad \text{for} \quad i = 1, 2, 3, \\ \psi_4(\xi) \equiv \frac{1}{2}|\xi|^2, \end{cases} \tag{1.4}$$

satisfying

$$\int_{\mathbf{R}^3} \psi_j(\xi) Q(h, g) d\xi = 0 \text{ for } \; j = 0, 1, 2, 3, 4.$$

In the following, we define an inner product in $\xi \in \mathbf{R}^3$ w.r.t. the local Maxwellian $\mathbf{M}$ as:

$$\langle h, g \rangle \equiv \int_{\mathbf{R}^3} \frac{1}{\mathbf{M}} h(\xi) g(\xi) d\xi,$$

for functions $h, \ g$ of ξ so that the above integral is well-defined. With respect to this inner product, the following functions spanning the space of macroscopic, i.e. fluid components of the solution, are pairwise orthogonal:

$$\begin{cases} \chi_0(\xi; \rho, u, \theta) \equiv \frac{1}{\sqrt{\rho}} \mathbf{M}, \\ \chi_i(\xi; \rho, u, \theta) \equiv \frac{\xi^i - u^i}{\sqrt{R\theta\rho}} \mathbf{M} \quad \text{for} \quad i = 1, 2, 3, \\ \chi_4(\xi; \rho, u, \theta) \equiv \frac{1}{\sqrt{6\rho}} \left(\frac{|\xi - u|^2}{R\theta} - 3 \right) \mathbf{M}, \\ \qquad \langle \chi_i, \chi_j \rangle = \delta_{ij}, \quad \text{for } i, j = 0, 1, 2, 3, 4. \end{cases} \tag{1.5}$$

By using these five functions, we define the macroscopic projection $\mathbf{P}_0$ and microscopic projection $\mathbf{P}_1$ as follows:

$$\begin{cases} \mathbf{P}_0 h \equiv \sum_{j=0}^{4} \langle h, \chi_j \rangle \chi_j, \\ \mathbf{P}_1 h \equiv h - \mathbf{P}_0 h. \end{cases} \tag{1.6}$$

Notice that the operators $\mathbf{P}_0$ and $\mathbf{P}_1$ are projections satisfying

$$\mathbf{P}_0 \mathbf{P}_0 = \mathbf{P}_0, \qquad \mathbf{P}_1 \mathbf{P}_1 = \mathbf{P}_1, \qquad \mathbf{P}_0 \mathbf{P}_1 = \mathbf{P}_1 \mathbf{P}_0 = 0.$$

A function $h(\xi)$ is called microscopic or non-fluid if it has no fluid components, that is,

$$\int_{\mathbf{R}^3} h(\xi) \psi_j(\xi) d\xi = 0, \text{ for } j = 0, 1, 2, 3, 4. \tag{1.7}$$

It is clear that such function is in the range of the microscopic projection $\mathbf{P}_1$. Under this decomposition, the solution $f(t, x, \xi)$ of the Boltzmann equation satisfies,

$$\mathbf{P}_0 f = \mathbf{M}, \qquad \mathbf{P}_1 f = \mathbf{G}.$$

Then by writing $f(t, x, \xi) = \mathbf{M}(t, x, \xi) + \mathbf{G}(t, x, \xi)$, the Boltzmann equation becomes:

$$(\mathbf{M} + \mathbf{G})_t + \xi_1 (\mathbf{M} + \mathbf{G})_x = \Big(2Q(\mathbf{G}, \mathbf{M}) + Q(\mathbf{G}, \mathbf{G}) \Big). \tag{1.8}$$

Based on this decomposition and (1.8), we now rewrite the Boltzmann equation as follows. First, as usual, a system of five conservation laws can be obtained by taking the inner product of the Boltzmann equation with the collision invariants $\psi_\alpha(\xi)$:

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = -\left(\int_{\mathbf{R}^3} \xi_1^2 \mathbf{G} d\xi\right)_x, \\ (\rho u_2)_t + (\rho u_1 u_2)_x = -\left(\int_{\mathbf{R}^3} \xi_1 \xi_2 \mathbf{G} d\xi\right)_x, \\ (\rho u_3)_t + (\rho u_1 u_3)_x = -\left(\int_{\mathbf{R}^3} \xi_1 \xi_3 \mathbf{G} d\xi\right)_x, \\ \left[\rho(\frac{1}{2}|u|^2 + \mathbf{E})\right]_t + \left(u_1\left(\rho\left(\frac{1}{2}|u|^2 + \mathbf{E}\right) + p\right)\right)_x = -\frac{1}{2}\left(\int_{\mathbf{R}^3} \xi_1 |\xi|^2 \mathbf{G} d\xi\right)_x. \end{cases} \tag{1.9}$$

Here p is the pressure for the monatomic gases:

$$p = \frac{2}{3}\rho\mathbf{E} = R\rho\theta.$$

Moreover, the microscopic equation for $\mathbf{G}$ is obtained by applying the microscopic projection $\mathbf{P}_1$ to (1.8):

$$\mathbf{G}_t + \mathbf{P}_1(\xi_1 \mathbf{G}_x + \xi_1 \mathbf{M}_x) = L_{\mathbf{M}}\mathbf{G} + Q(\mathbf{G}, \mathbf{G}), \tag{1.10}$$

i.e.,

$$\begin{aligned} \mathbf{G} &= L_{\mathbf{M}}^{-1}(\xi_1 \mathbf{M}_x) + L_{\mathbf{M}}^{-1}\Big(\mathbf{G}_t + \mathbf{P}_1(\xi_1 \mathbf{G}_x) - Q(\mathbf{G}, \mathbf{G})\Big) \\ &:= L_{\mathbf{M}}^{-1}(\xi_1 \mathbf{M}_x) + \Theta, \end{aligned} \tag{1.11}$$

where

$$L_{\mathbf{M}} g = L_{[\rho,u,\theta]} g \equiv Q\Big(\mathbf{M} + g, \mathbf{M} + g\Big) - Q(g, g).$$

Now plug (1.11) into (1.9), we have another form of Boltzmann equation which is in the type of fluid dynamics.

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = -\left(\int_{\mathbf{R}^3} \xi_1^2 L_{\mathbf{M}}^{-1}(\xi_1 \mathbf{M}_x) d\xi\right)_x - \left(\int_{\mathbf{R}^3} \xi_1^2 \Theta d\xi\right)_x, \\ (\rho u_2)_t + (\rho u_1 u_2)_x = -\left(\int_{\mathbf{R}^3} \xi_1 \xi_2 L_{\mathbf{M}}^{-1}(\xi_1 \mathbf{M}_x) d\xi\right)_x - \left(\int_{\mathbf{R}^3} \xi_1 \xi_2 \Theta d\xi\right)_x, \\ (\rho u_3)_t + (\rho u_1 u_3)_x = -\left(\int_{\mathbf{R}^3} \xi_1 \xi_3 L_{\mathbf{M}}^{-1}(\xi_1 \mathbf{M}_x) d\xi\right)_x - \left(\int_{\mathbf{R}^3} \xi_1 \xi_3 \Theta d\xi\right)_x, \\ \left[\rho(\frac{1}{2}|u|^2 + \mathbf{E})\right]_t + \left(u_1\left(\rho\left(\frac{1}{2}|u|^2 + \mathbf{E}\right) + p\right)\right)_x \\ \qquad = -\frac{1}{2}\left(\int_{\mathbf{R}^3} \xi_1 |\xi|^2 L_{\mathbf{M}}^{-1}(\xi_1 \mathbf{M}_x) d\xi\right)_x - \frac{1}{2}\left(\int_{\mathbf{R}^3} \xi_1 |\xi|^2 \Theta d\xi\right)_x. \end{cases} \tag{1.12}$$

Notice that if one neglects all the terms containing Θ, (1.12) is exactly the Navier-Stokes equations. By rewriting the Boltzmann equation in this form, we can later perform the analysis used for conservation laws to get the desired estimates.

For later use, we now recall some properties of the linearized collision operator $L_{\mathbf{M}}$. By definition, $L_{\mathbf{M}}$ is symmetric:

$$\langle h, L_{\mathbf{M}} g \rangle = \langle L_{\mathbf{M}} h, g \rangle,$$

and the null space N of $L_{\mathbf{M}}$ contains only the macroscopic variables:

$$\chi_j, \qquad j = 0, \cdots, 4.$$

For the hard sphere model, $L_{\mathbf{M}}$ takes the form, cf. [14, 13]

$$(L_{\mathbf{M}} h)(\xi) = -\nu(\xi; \rho, u, \theta) h(\xi) + \sqrt{\mathbf{M}(\xi)} K_{\mathbf{M}} \left(\left(\frac{h}{\sqrt{\mathbf{M}}} \right)(\xi) \right). \tag{1.13}$$

Here $K_{\mathbf{M}}(\cdot) = K_{1\mathbf{M}}(\cdot) + K_{2\mathbf{M}}(\cdot)$ is a symmetric compact L^2-operator. And $\nu(\xi; \rho, u, \theta)$ and $K_{i\mathbf{M}}(\cdot)$ have the following expressions

$$\begin{cases} \nu(\xi; \rho, u, \theta) = \frac{2\rho}{\sqrt{2\pi R\theta}} \left\{ \left(\frac{R\theta}{|\xi-u|} + |\xi - u| \right) \int_0^{|\xi-u|} \exp\left(-\frac{y^2}{2R\theta}\right) dy + R\theta \exp\left(-\frac{|\xi-u|^2}{2R\theta}\right) \right\}, \\ k_{1\mathbf{M}}(\xi, \xi_*) = \frac{\pi\rho}{\sqrt{(2\pi R\theta)^3}} |\xi - \xi_*| \exp\left(-\frac{|\xi-u|^2}{4R\theta} - \frac{|\xi_*-u|^2}{4R\theta}\right), \\ k_{2\mathbf{M}}(\xi, \xi_*) = \frac{2\rho}{\sqrt{2\pi R\theta}} |\xi - \xi_*|^{-1} \exp\left(-\frac{|\xi-\xi_*|^2}{8R\theta} - \frac{(|\xi|^2-|\xi_*|^2)^2}{8R\theta|\xi-\xi_*|^2}\right), \end{cases}$$

where $k_{i\mathbf{M}}(\xi, \xi_*)$ is the kernal of the operator $K_{i\mathbf{M}}$ respectively. Furthermore, there exists $\sigma_0(\rho, u, \theta) > 0$ such that for any function $h(\xi) \in N$

$$\langle h, L_{\mathbf{M}} h \rangle \leq -\sigma_0(\rho, u, \theta) \langle h, h \rangle,$$

which implies cf. [13]

$$\langle h, L_{\mathbf{M}} h \rangle \leq -\sigma(\rho, u, \theta) \langle (1 + |\xi|) h, h \rangle \tag{1.14}$$

with some constant $\sigma(\rho, u, \theta) > 0$.

For later use, notice also that the projections $\mathbf{P}_0$ and $\mathbf{P}_1$ have the following basic properties:

$$\begin{cases} \mathbf{P}_0(\psi_j \mathbf{M}) = \psi_j \mathbf{M}, \ \mathbf{P}_1(\psi_j \mathbf{M}) = 0, \ j = 0, 1, 2, 3, 4, \\ L_{\mathbf{M}} \mathbf{P}_1 = \mathbf{P}_1 L_{\mathbf{M}} = L_{\mathbf{M}}, \ \mathbf{P}_1(Q(h, h)) = Q(h, h), \\ L_{\mathbf{M}} \mathbf{P}_0 = \mathbf{P}_0 L_{\mathbf{M}} = 0, \ \mathbf{P}_0(Q(h, h)) = 0, \\ \langle \psi_j \mathbf{M}, h \rangle = \langle \psi_j \mathbf{M}, \mathbf{P}_0 h \rangle, \ j = 0, 1, 2, 3, 4, \\ \langle h, L_{\mathbf{M}} g \rangle = \langle \mathbf{P}_1 h, L_{\mathbf{M}}(\mathbf{P}_1 g) \rangle, \\ \left\langle h, L_{\mathbf{M}}^{-1}(\mathbf{P}_1 g) \right\rangle = \left\langle L_{\mathbf{M}}^{-1}(\mathbf{P}_1 h), \mathbf{P}_1 g \right\rangle = \left\langle \mathbf{P}_1 h, L_{\mathbf{M}}^{-1}(\mathbf{P}_1 g) \right\rangle. \end{cases}$$

Now we come back to the problem considered in this paper, that is, the time asymptotic behavior of global smooth solution $f(t, x, \xi)$ of the Boltzmann equation (1.1) with initial data as a small perturbation of a nonlinear wave pattern consisting two rarefaction waves. Let

$$f(t, x, \xi)|_{t=0} = f_0(x, \xi) \to \begin{cases} \mathbf{M}_l \equiv \mathbf{M}_{[\rho_l, u_l, \theta_l]} = \frac{\rho_l}{\sqrt{(2\pi R\theta_l)^3}} \exp\left(-\frac{|\xi-u_l|^2}{2R\theta_l}\right), \quad x \to -\infty, \\ \mathbf{M}_r \equiv \mathbf{M}_{[\rho_r, u_r, \theta_r]} = \frac{\rho_r}{\sqrt{(2\pi R\theta_r)^3}} \exp\left(-\frac{|\xi-u_r|^2}{2R\theta_r}\right), \quad x \to +\infty. \end{cases} \tag{1.15}$$

Here $\rho_l, \theta_l, \rho_r, \theta_r > 0, u_l = (u_{1l}, 0, 0), u_r = (u_{1r}, 0, 0)$ are constants. Assume the following Riemann problem for compressible Euler equations

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = 0, \\ (\rho u_2)_t + (\rho u_1 u_2)_x = 0, \\ (\rho u_3)_t + (\rho u_1 u_3)_x = 0, \\ \left[\rho(\frac{1}{2}|u|^2 + \mathbf{E})\right]_t + \left(u_1\left(\rho\left(\frac{1}{2}|u|^2 + \mathbf{E}\right) + p\right)\right)_x = 0, \end{cases} \tag{1.16}$$

$$(\rho, u, \theta)(t,x)|_{t=0} = (\rho_0^r, u_0^r, \theta_0^r)(x) = \begin{cases} (\rho_l, u_l, \theta_l), & x < 0 \\ (\rho_r, u_r, \theta_r), & x > 0 \end{cases} \tag{1.17}$$

admits a unique global weak solution $\left(\rho^R(t,x), u^R(t,x), \theta^R(t,x)\right)$ containing a rarefaction wave of the first family, denoted by $\left(\rho^{R_1}(t,x), u^{R_1}(t,x), \theta^{R_1}(t,x)\right)$, and another of the third family, denoted by $\left(\rho^{R_3}(t,x), u^{R_3}(t,x), \theta^{R_3}(t,x)\right)$ with $u_i^R = u_i^{R_1} = u_i^{R_3} = 0, i = 2, 3$. That is, there exists a unique constant state $(\rho_m, u_m, \theta_m) \in \mathbf{R}^5 (\rho_m > 0, \theta_m > 0, u_{im} = 0, i = 2, 3)$ such that $(\rho_m, u_m, \theta_m) \in R_1(\rho_l, u_l, \theta_l)$ and $(\rho_r, u_r, \theta_r) \in R_3(\rho_m, u_m, \theta_m)$. Here

$$\begin{cases} R_1(\rho_l, u_l, \theta_l) = \Big\{ (\rho, u, \theta) \Bigm| S = \overline{S},\ u_1 + \sqrt{15k}\rho^{\frac{1}{3}} \exp\left(\frac{S}{2}\right) = u_{1l} + \sqrt{15k}\rho_l^{\frac{1}{3}} \exp\left(\frac{\overline{S}}{2}\right), \\ \qquad u_2 = u_3 = 0,\ u_1 > u_{1l},\ \rho < \rho_l \Big\}, \\ R_3(\rho_m, u_m, \theta_m) = \Big\{ (\rho, u, \theta) \Bigm| S = \overline{S},\ u_1 - \sqrt{15k}\rho^{\frac{1}{3}} \exp\left(\frac{S}{2}\right) = u_{1m} - \sqrt{15k}\rho_m^{\frac{1}{3}} \exp\left(\frac{\overline{S}}{2}\right), \\ \qquad u_2 = u_3 = 0,\ u_1 > u_{1m},\ \rho < \rho_m \Big\}, \end{cases} \tag{1.18}$$

where

$$\begin{cases} S = -\frac{2}{3}\ln\rho + \ln(2\pi R\theta) + 1 = -\frac{2}{3}\ln\rho_l + \ln(2\pi R\theta_l) + 1 \\ \quad = -\frac{2}{3}\ln\rho_r + \ln(2\pi R\theta_r) + 1 \equiv \overline{S}, \\ k = \frac{1}{2\pi e}. \end{cases}$$

In other words,

$$\left(\rho^R, u^R, \theta^R\right)(t,x) = \left(\rho^{R_1} + \rho^{R_3} - \rho_m, u^{R_1} + u^{R_3} - u_m, \theta^{R_1} + \theta^{R_3} - \theta_m\right)(t,x). \tag{1.19}$$

Similar to the corresponding work on the Navier-Stokes equations, cf. [21], we will show that if the initial data is a small perturbation of a Maxwellian state defined by a smooth approximation of the above Riemann solution (1.19), then the solution to the Boltzmann eqaution $f(t,x,\xi)$ tends to $\mathbf{M}_{[\rho^R(t,x), u^R(t,x), \theta^R(t,x)]}$ as $t \to +\infty$.

Now we give a brief description of the approximate rarefaction waves. Given a suitably small but fixed constant $\varepsilon > 0$, let $w_i(t,x)$ be the unique global smooth solution to the following Cauchy problem of the Burgers' equation

$$\begin{cases} w_{it} + w_i w_{ix} = 0, \\ w_i(t,x)|_{t=0} = w_{i0}(x) = \frac{1}{2}(w_{i+} + w_{i-}) + \frac{1}{2}(w_{i+} - w_{i-})\tanh(\varepsilon x), \quad i = 1, 3, \end{cases} \tag{1.20}$$

where

$$\begin{cases} w_{1-} = \lambda_1(\rho_l, u_l, \theta_l) = u_{1l} - \frac{\sqrt{15k}}{3}\rho_l^{\frac{1}{3}} \exp\left(\frac{S}{2}\right), \\ w_{1+} = \lambda_1(\rho_m, u_m, \theta_m) = u_{1m} - \frac{\sqrt{15k}}{3}\rho_m^{\frac{1}{3}} \exp\left(\frac{S}{2}\right), \\ w_{3-} = \lambda_3(\rho_m, u_m, \theta_m) = u_{1m} + \frac{\sqrt{15k}}{3}\rho_m^{\frac{1}{3}} \exp\left(\frac{S}{2}\right), \\ w_{3+} = \lambda_3(\rho_r, u_r, \theta_r) = u_{1r} + \frac{\sqrt{15k}}{3}\rho_r^{\frac{1}{3}} \exp\left(\frac{S}{2}\right), \end{cases} \tag{1.21}$$

then, the approximation of the rarefaction waves profile $\left(\overline{\rho}(t,x), \overline{u}(t,x), \overline{\theta}(t,x)\right)$ is defined by

$$\left(\overline{\rho}, \overline{u}, \overline{\theta}\right)(t,x) = \left(\rho^{A_1} + \rho^{A_3} - \rho_m, u^{A_1} + u^{A_3} - u_m, \theta^{A_1} + \theta^{A_3} - \theta_m\right)(t + t_0, x), \tag{1.22}$$

where $t_0 = \frac{1}{d_1\varepsilon^2} > 0$, $d_1 = \frac{\sqrt{15k}}{3}\rho_m^{\frac{1}{3}} \exp\left(\frac{S}{2}\right) > 0$, and $\left(\rho^{A_i}, u^{A_i}, \theta^{A_i}\right)(t,x)(i = 1,3)$ are given by the following equations,

$$\begin{cases} u_1^{A_i}(t,x) + (-1)^{\frac{1+i}{2}}\frac{\sqrt{15k}}{3}\left(\rho^{A_i}(t,x)\right)^{\frac{1}{3}} \exp\left(\frac{S}{2}\right) = w_i(t,x),\ i = 1,3, \\ u_1^{A_1}(t,x) + \sqrt{15k}\left(\rho^{A_1}(t,x)\right)^{\frac{1}{3}} \exp\left(\frac{S}{2}\right) = u_{1l} + \sqrt{15k}\rho_l^{\frac{1}{3}} \exp\left(\frac{S}{2}\right), \\ u_1^{A_3}(t,x) - \sqrt{15k}\left(\rho^{A_3}(t,x)\right)^{\frac{1}{3}} \exp\left(\frac{S}{2}\right) = u_{1r} - \sqrt{15k}\rho_r^{\frac{1}{3}} \exp\left(\frac{S}{2}\right), \\ \theta^{A_i}(t,x) = \frac{3}{2}k\left(\rho^{A_i}(t,x)\right)^{\frac{2}{3}} \exp(\overline{S}),\ u_2^{A_i} = u_3^{A_i} = 0,\ i = 1,3. \end{cases} \tag{1.23}$$

Based on the decomposed system (1.12), the stability of a global Maxwellian state was proved in [18] by energy method. The analysis there relies on the microscopic and macroscopic $H-$theorems with respect to a global Maxwellian $\mathbf{M}_- \equiv \mathbf{M}_{[\rho_-, u_-, \theta_-]}$ because the initial perturbation is a small perturbation of this global Maxwellian. However, we are now considering the perturbation of a nonlinear wave pattern which may not be small, the energy estimates w.r.t. a global Maxwellian state is not enough. Therefore, a combination of the energy estimates w.r.t. a global and the local Maxwellian states will be used. For this reason, another form of the microscopic H-theorem is needed to relate the dissipation estimates with different weights. In fact, we will show in Section 2 that there exists a positive constant (which is not necessary to be small)

$$\eta_0 = \eta_0(\rho, u, \theta; \rho_-, u_-, \theta_-) > 0$$

such that if $\frac{\theta}{2} < \theta_- < \theta$ and $|\rho - \rho_-| + |u - u_-| + |\theta - \theta_-| < \eta_0$, the following microscopic $H-$theorem

$$-\int_{\mathbf{R}^3} \frac{GL_{\mathbf{M}}G}{\mathbf{M}_-} d\xi \geq \overline{\sigma} \int_{\mathbf{R}^3} \frac{(1 + |\xi|)G^2}{\mathbf{M}_-} d\xi, \tag{1.24}$$

holds for some positive constant $\overline{\sigma} = \overline{\sigma}(\rho, u, \theta; \rho_-, u_-, \theta_-) > 0$.

With the above notations, we can now state the main result in this paper as follows.

Theorem 1.1 *Let the approximate rarefaction wave* $\left(\overline{\rho}, \overline{u}, \overline{\theta}\right)(t,x)$ *be defined in (1.22). If*

$$\begin{cases} \delta = \max\left\{|\rho_l - \rho_m| + |u_l - u_m| + |\theta_l - \theta_m|, |\rho_r - \rho_m| + |u_r - u_m| + |\theta_r - \theta_m|\right\} < \eta_0, \\ \frac{1}{2} \sup_{(t,x) \in \mathbf{R}_+ \times \mathbf{R}} \overline{\theta}(t,x) < \inf_{(t,x) \in \mathbf{R}_+ \times \mathbf{R}} \overline{\theta}(t,x), \end{cases}$$

(1.25)

There exists a global Maxwellian $\mathbf{M}_-$ *and sufficiently small positive constants* $\varepsilon_0, \varepsilon$ *such that the following holds. Let the initial data* $f_0(x,\xi)$ *satisfy*

$$\left\| f_0(x,\xi) - \mathbf{M}_{[\bar{\rho}(0,x),\bar{u}(0,x),\bar{\theta}(0,x)]} \right\|_{H_x^s\left(L_\xi^2\left(\frac{1}{\sqrt{\mathbf{M}_-}}\right)\right)} \leq \varepsilon_0, \tag{1.26}$$

then the Cauchy problem (1.1), (1.15) admits a unique global solution $f(t,x,\xi)$ *satisfying*

$$\left\| f(t,x,\xi) - \mathbf{M}_{[\bar{\rho},\bar{u},\bar{\theta}]} \right\|_{H_x^s\left(L_\xi^2\right)} \leq C_1\left(\varepsilon_0 + \varepsilon^{\frac{1}{16}}\right), \tag{1.27}$$

for some positive constant C_1*, and*

$$\lim_{t\to\infty} \left\| f(t,x,\xi) - \mathbf{M}_{[\rho^R,u^R,\theta^R]} \right\|_{L_x^\infty\left(L_\xi^2\right)} = 0. \tag{1.28}$$

Here the constant ε *comes from the definition of the approximate rarefaction wave.* $f(\xi) \in L_\xi^2\left(\frac{1}{\sqrt{\mathbf{M}_-}}\right)$ *means that* $\frac{f(\xi)}{\sqrt{\mathbf{M}_-}} \in L_\xi^2$*, and* $\mathbf{M}_- = \mathbf{M}_{[\rho_-,u_-,\theta_-]}$ *is a global Maxwellian satisfying* $\frac{1}{2}\theta(t,x) < \theta_- < \theta(t,x)$*, and* $|\rho(t,x)-\rho_-| + |u(t,x)-u_-| + |\theta(t,x)-\theta_-| < \eta_0$ *for all* $(t,x) \in \mathbf{R}_+ \times \mathbf{R}$.

Remark 1.1 *Even though the initial data* $f_0(x,\xi)$ *is assumed to be a small perturbation of a nonlinear wave pattern, the strengths of the nonlinear waves need not to be sufficiently small. The condition on the strengths of the rarefaction waves comes from (1.25) where* η_0 *depends on the first eigenvalue of the linearized operator* $L_{\mathbf{M}}$ *and the values of* (ρ,u,θ)*,* (ρ_-,u_-,θ_-).

The main idea of the proof of Theorem 1.1 can be outlined as follows. First, the energy estimate w.r.t. the local Maxwellian is necessary because the nonlinear wave background profile is not sufficiently small. In this case, the microscopic $H-$theorem, i.e. (1.14) and (1.24) has dissipation on the microscopic component of the order of $\int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{(1+|\xi|)|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau$ where the order of ξ is only 1. However, the energy estimate by using the weight of local Maxwellian has error terms with a polynomial of ξ with order greater than 1 because of the derivatives on the local Maxwellian. Hence, another set of energy estimates based on a global Maxwellian chosen suitably is needed to close the analysis. Therefore, we first perform the energy estimates w.r.t. the local Maxwellian $\mathbf{M}$. Notice that a typical error term $\int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}} \mathbf{M}_t d\xi dx d\tau$ appears and satisfies

$$\left| \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}} \mathbf{M}_t d\xi dx d\tau \right| \leq C\left(\theta-\theta_-, \rho, u, \rho_-, u_-\right)(\varepsilon+\delta_0) \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau,$$

where the error term is now an integral with weight $\mathbf{M}_-$ and a small factor of order of $\varepsilon + \delta_0$.

The second step is to perform the energy estimate by using the global Maxwellian $\mathbf{M}_-$. In this case, although an additional terms in the form of integrals of the fluid components and their derivatives appear because the orthogonality property of $\mathbf{M}$ and $\mathbf{G}$ fails w.r.t. weight $\mathbf{M}_-$, the small factor $\varepsilon + \delta_0$ helps to yield the desired estimates. It is worth to pointing out that to get the higher order energy estimates on the macroscopic component

$\mathbf{M}$, there is no need for the use of global Maxwellian $\mathbf{M}_-$ because all polynomials of ξ (if any) can be absorbed by the local Maxwellian $\mathbf{M}$.

Before concluding this section, we should mention that there have been extensive study on Boltzmann equation which gives arise to rich physical and solution phenomena, such as Knudsen layer, ghost effects, incompressible flow limit etc. Since these are beyond the scope of this paper, we will not refer them here. Before the energy method based on the decomposition (1.12) is used, the elegant analysis using the spectral properties of the linearized collision operator $L_{\mathbf{M}}$ has been used to obtained existence and large time behavior of solutions to Boltzmann equation, see [16, 23, 25] and references therein.

This rest of this paper is arranged as follows: a microscopic and a macroscopic $H-$theorems will be given in Section 2. Some properties of the smooth approximation to rarefaction wave solutions are presented in Section 3. And the proof of Theorem 1.1 will be given in Section 4 for the case when $s = 5$. The case when $s > 5$ can be discussed similarly.

2 $H-$Theorem

The $H-$theorem of the Boltzmann equation is based on the special property of the bilinear structure of $Q(f,f)$ which satisfies

$$\int_{\mathbf{R}^3} Q(f,f)\ln f d\xi \le 0,$$

and the equality holds only when the solution $f(t,x,\xi)$ is a Maxwellian, i.e. $f(t,x,\xi) = \mathbf{M}_{[\rho(t,x),u(t,x),\theta(t,x)]}$. According to the dissipative effects on the macroscopic and microscopic components, the H-theorem can be viewed from two aspects. The first one is mainly on the linearized collision operator $L_{\mathbf{M}}$ acting on the microscopic components stated in (1.14) and (1.24) which is called the microscopic $H-$theorem. The second one comes from the nonlinear collision operator where the viscosity and heat conductivity can be expressed.

In the following, we will first give a proof of (1.24). To do this, we need a lemma from [11].

Lemma 2.1 *Assume that $-2 \le \beta \le 0$, then there exists a positive constant $C_2 > 0$ such that*

$$\int_{\mathbf{R}^3} \frac{(1+|\xi|)^\beta Q(f,g)^2}{\mathbf{M}} d\xi \le \frac{C_2}{2}\left\{\int_{\mathbf{R}^3} \frac{(1+|\xi|)^{\beta+2} f^2}{\mathbf{M}} d\xi \cdot \int_{\mathbf{R}^3} \frac{g^2}{\mathbf{M}} d\xi + \int_{\mathbf{R}^3} \frac{f^2}{\mathbf{M}} d\xi \cdot \int_{\mathbf{R}^3} \frac{(1+|\xi|)^{\beta+2} g^2}{\mathbf{M}} d\xi\right\}, \quad (2.1)$$

where $\mathbf{M}$ can be any Maxwellian so that the above integrals are well defined.

Notice for the case of hard sphere model, only $\beta = -1$ is used here.

Lemma 2.2 *If $\frac{\theta}{2} < \theta_- < \theta$, then there exist two positive constants $\overline{\sigma} = \overline{\sigma}(\rho,u,\theta;\rho_-,u_-,\theta_-)$ and $\eta_0 = \eta_0(\rho,u,\theta;\rho_-,u_-,\theta_-)$ such that if $|\rho-\rho_-|+|u-u_-|+|\theta-\theta_-| < \eta_0$, we have for $h(\xi) \in N$,*

$$-\int_{\mathbf{R}^3} \frac{h L_{\mathbf{M}} h}{\mathbf{M}_-} d\xi \ge \overline{\sigma} \int_{\mathbf{R}^3} \frac{(1+|\xi|)h^2}{\mathbf{M}_-} d\xi.$$

Proof. By Lemma 2.1, we have

$$
\begin{aligned}
&-\int_{\mathbf{R}^3}\tfrac{hL_{\mathbf{M}}h}{\mathbf{M}_-}d\xi = -\int_{\mathbf{R}^3}\tfrac{hL_{\mathbf{M}_-}h}{\mathbf{M}_-}d\xi + 2\int_{\mathbf{R}^3}\tfrac{hQ(\mathbf{M}_- - \mathbf{M},h)}{\mathbf{M}_-}d\xi\\
&\geq \sigma(\rho_-,u_-,\theta_-)\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)h^2}{\mathbf{M}_-}d\xi + 2\int_{\mathbf{R}^3}\tfrac{hQ(\mathbf{M}_- - \mathbf{M},h)}{\mathbf{M}_-}d\xi\\
&\geq \tfrac{3}{4}\sigma(\rho_-,u_-,\theta_-)\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)h^2}{\mathbf{M}_-}d\xi - \tfrac{4}{\sigma(\rho_-,u_-,\theta_-)}\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)^{-1}Q(\mathbf{M}_- - \mathbf{M},h)^2}{\mathbf{M}_-}d\xi\\
&\geq \tfrac{3}{4}\sigma(\rho_-,u_-,\theta_-)\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)h^2}{\mathbf{M}_-}d\xi - \tfrac{4C_2}{\sigma(\rho_-,u_-,\theta_-)}\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)(\mathbf{M}_- - \mathbf{M})^2}{\mathbf{M}_-}d\xi\cdot\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)h^2}{\mathbf{M}_-}d\xi.
\end{aligned}
\tag{2.2}
$$

Since $\frac{\theta}{2} < \theta_- < \theta$, we can choose a positive constant $C_3 = C_3(\rho,u,\theta;\rho_-,u_-,\theta_-)$ such that

$$
\int_{|\xi|\geq C_3}\frac{(1+|\xi|)(\mathbf{M}_- - \mathbf{M})^2}{\mathbf{M}_-}d\xi \leq \frac{\sigma(\rho_-,u_-,\theta_-)^2}{16C_2}. \tag{2.3}
$$

On the other hand,

$$
\int_{|\xi|\leq C_3}\frac{(1+|\xi|)(\mathbf{M}_- - \mathbf{M})^2}{\mathbf{M}_-}d\xi \leq \frac{C_4}{4C_2}\Big(|\rho-\rho_-|+|u-u_-|+|\theta-\theta_-|\Big)^2 \tag{2.4}
$$

holds for some positive constant $C_4 = C_4(\rho,u,\theta;\rho_-,u_-,\theta_-)$.

By putting (2.3) and (2.4) into (2.2), and choosing

$$
\eta_0 = \frac{\sigma(\rho_-,u_-,\theta_-)}{\sqrt{2C_4(\rho,u,\theta;\rho_-,u_-,\theta_-)}}.
$$

(2.2) gives the statement of the lemma.

The following is a direct corollary of Lemma 2.2 and the Cauchy inequality.

Corollary 2.1 *Under the assumptions in Lemma 2.2, we have*

$$
\begin{cases}
\int_{\mathbf{R}^3}\frac{1+|\xi|}{\mathbf{M}}\left|L_{\mathbf{M}}^{-1}h\right|^2 d\xi \leq \sigma^{-2}\int_{\mathbf{R}^3}\frac{(1+|\xi|)^{-1}h^2(\xi)}{\mathbf{M}}d\xi,\\
\int_{\mathbf{R}^3}\frac{1+|\xi|}{\mathbf{M}_-}\left|L_{\mathbf{M}}^{-1}h\right|^2 d\xi \leq \overline{\sigma}^{-2}\int_{\mathbf{R}^3}\frac{(1+|\xi|)^{-1}h^2(\xi)}{\mathbf{M}_-}d\xi
\end{cases}
\tag{2.5}
$$

holds for each $h(\xi)\in N$.

For later use, we also include the following lemma from [18].

Lemma 2.3 *Under the conditions in Lemma 2.2, there exists a constant $C_5 > 0$ such that for each $k\in\mathbf{Z}_+$ and constant $\lambda > 0$ we have*

$$
\left|\int_{\mathbf{R}^3}\frac{g_1\mathbf{P}_1(|\xi|^k g_2)}{\mathbf{M}_-}d\xi - \int_{\mathbf{R}^3}\frac{g_1|\xi|^k g_2}{\mathbf{M}_-}d\xi\right| \leq C_5\int_{\mathbf{R}^3}\frac{\lambda|g_1|^2+\lambda^{-1}|g_2|^2}{\mathbf{M}_-}d\xi. \tag{2.6}
$$

In order to study the nonlinear wave behavior of the solutions, we also need to use the macroscopic version of the $H-$theorem studied in [18]. To be self-contained, we include it as follows. Set the macroscopic entropy S by

$$
-\frac{3}{2}\rho S \equiv \int_{\mathbf{R}^3}\mathbf{M}\ln\mathbf{M}d\xi. \tag{2.7}
$$

Direct calculation yields

$$-\frac{3}{2}(\rho S)_t-\frac{3}{2}(\rho u_1 S)_x+\left(\int_{\mathbf{R}^3}(\xi_1\ln\mathbf{M})\mathbf{G}d\xi\right)_x=\int_{\mathbf{R}^3}\frac{\mathbf{G}\xi_1\partial_x\mathbf{M}}{\mathbf{M}}d\xi \tag{2.8}$$

and

$$\begin{cases} S=-\frac{2}{3}\ln\rho+\ln(2\pi R\theta)+1,\\ p=\frac{2}{3}\rho\theta=k\rho^{\frac{5}{3}}\exp(S),\\ \mathbf{E}=\theta,\quad R=\frac{2}{3}.\end{cases} \tag{2.9}$$

An convex entropy-entropy flux pair (η,q) around a Maxwellian $\overline{\mathbf{M}}=\mathbf{M}_{[\overline{\rho},\overline{u},\overline{\theta}]}$ ($\overline{u}_i=0,i=2,3$) can be given as follows. Denote the conservation laws (1.9) by

$$\mathbf{m}_t+\mathbf{n}_x=-\begin{pmatrix}0\\ \int_{\mathbf{R}^3}\xi_1^2\mathbf{G}d\xi\\ \int_{\mathbf{R}^3}\xi_1\xi_2\mathbf{G}d\xi\\ \int_{\mathbf{R}^3}\xi_1\xi_3\mathbf{G}d\xi\\ \frac{1}{2}\int_{\mathbf{R}^3}\xi_1|\xi|^2\mathbf{G}d\xi\end{pmatrix}_x. \tag{2.10}$$

Here

$$\begin{cases}\mathbf{m}=(m_0,m_1,m_2,m_3,m_4)^t=\left(\rho,\rho u_1,\rho u_2,\rho u_3,\rho\left(\frac{1}{2}|u|^2+\theta\right)\right)^t,\\ \mathbf{n}=(n_0,n_1,n_2,n_3,n_4)^t=\left(\rho u_1,\rho u_1^2+\frac{2}{3}\rho\theta,\rho u_1u_2,\rho u_1u_3,\rho u_1\left(\frac{1}{2}|u|^2+\frac{5}{3}\theta\right)\right)^t.\end{cases}$$

Now we define an entropy-entropy flux pair (η,q) as

$$\begin{cases}\eta=\overline{\theta}\left\{-\frac{3}{2}\rho S+\frac{3}{2}\overline{\rho}\overline{S}+\frac{3}{2}\nabla_{\mathbf{m}}(\rho S)|_{\mathbf{m}=\overline{\mathbf{m}}}(\mathbf{m}-\overline{\mathbf{m}})\right\},\\ q=\overline{\theta}\left\{-\frac{3}{2}\rho u_1S+\frac{3}{2}\overline{\rho}\overline{u}_1\overline{S}+\frac{3}{2}\nabla_{\mathbf{m}}(\rho S)|_{\mathbf{m}=\overline{\mathbf{m}}}(\mathbf{n}-\overline{\mathbf{n}})\right\}.\end{cases} \tag{2.11}$$

Since

$$\begin{cases}(\rho S)_{m_0}=S+\frac{|u|^2}{2\theta}-\frac{5}{3},\\ (\rho S)_{m_i}=-\frac{u_i}{\theta},\ i=1,2,3,\\ (\rho S)_{m_4}=\frac{1}{\theta},\end{cases}$$

we have

$$\begin{cases}\eta=\frac{3}{2}\left\{\rho\theta-\overline{\theta}\rho S+\rho\left[\left(\overline{S}-\frac{5}{3}\right)\overline{\theta}+\frac{|u-\overline{u}|^2}{2}\right]+\frac{2}{3}\overline{\rho}\overline{\theta}\right\},\\ q=u_1\eta+(u_1-\overline{u}_1)\left(\rho\theta-\overline{\rho}\overline{\theta}\right).\end{cases} \tag{2.12}$$

The entropy-entropy flux thus construct have the following properties. First, $\eta(\overline{\mathbf{m}})=0,\nabla_{\mathbf{m}}\eta(\overline{\mathbf{m}})=0$. Then the Hessian $\frac{\partial^2\eta}{\partial m_i\partial m_j}$ equals to

$$\frac{3\overline{\theta}}{2\rho\theta^2}\begin{pmatrix}\frac{5}{3}\theta^2+\frac{1}{4}|u|^4 & -\frac{1}{2}u_1|u|^2 & -\frac{1}{2}u_2|u|^2 & -\frac{1}{2}u_3|u|^2 & \frac{1}{2}|u|^2-\theta\\ -\frac{1}{2}u_1|u|^2 & \theta+u_1^2 & u_1u_2 & u_1u_3 & -u_1\\ -\frac{1}{2}u_2|u|^2 & u_1u_2 & \theta+u_2^2 & u_2u_3 & -u_2\\ -\frac{1}{2}u_3|u|^2 & u_1u_3 & u_2u_3 & \theta+u_3^2 & -u_3\\ \frac{1}{2}|u|^2-\theta & -u_1 & -u_2 & -u_3 & 1\end{pmatrix},$$

which is positive definite for any $\mathbf{m}$ satisfying $\rho,\ \theta>0$. Thus for $\mathbf{m}$ in any closed bounded region in $\Sigma=\{\mathbf{m}:\rho>0,\theta>0\}$, there exists a positive constant C_6 such that

$$C_6^{-1}\left|\mathbf{m}-\overline{\mathbf{m}}\right|^2\leq\eta\leq C_6\left|\mathbf{m}-\overline{\mathbf{m}}\right|^2. \tag{2.13}$$

3 Approximate rarefaction waves

In this section, we present the approximate rarefaction waves solution as the background nonlinear profile to Boltzmann equation through the Riemann problem for Burgers' equation. The approximate rarefaction waves solution given below is from [21] where the stability of rarefaction waves for Navier-Stokes equations is studied. Here, we include its definition and L^p estimates for the convenience of the readers.

For $i=1,3$, consider

$$\begin{cases} w_t^{R_i}+w^{R_i}w_x^{R_i}=0,\\ w^{R_i}(0,x)=w_0^{R_i}(x)=\begin{cases} w_{i-}, & x<0\\ w_{i+}, & x>0\end{cases}\end{cases} \tag{3.1}$$

with $w_{i-}\leq w_{i+}$, which have continuous weak solutions of the form $w^{R_i}\left(\frac{x}{t}\right)$ given by

$$w^{R_i}(\xi)=\begin{cases} w_{i-}, & \xi\leq w_{i-},\\ \xi, & w_{i-}\leq\xi\leq w_{i+},\\ w_{i+}, & \xi\geq w_{i+}.\end{cases} \tag{3.2}$$

In [21], it is shown that $w^{R_i}\left(\frac{x}{t}\right)$ is approximated by the solution to the Cauchy problem (1.20) when t is sufficiently large which is summarized in the following lemmas.

Lemma 3.1 *The Cauchy problem (1.20) has a unique global smooth solution $w_i(t,x)$ satisfying the following*

(i) $w_{i-}<w_i(t,x)<w_{i+},\ w_{ix}(t,x)>0,\ \forall(t,x)\in\mathbf{R}_+\times\mathbf{R}$;

(ii) For any $p(1\leq p\leq\infty)$, there exists a constant $C(p)$, depending only on p, such that

$$\begin{cases} \left\|w_{ix}(t,x)\right\|_{L^p}\leq C(p)\min\left\{\delta_i\varepsilon^{1-\frac{1}{p}},\ \delta_i^{\frac{1}{p}}t^{-1+\frac{1}{p}}\right\},\\ \left\|\frac{\partial^j}{\partial x^j}w_i(t,x)\right\|_{L^p}\leq C(p)\min\left\{\delta_i\varepsilon^{j-\frac{1}{p}},\ \varepsilon^{j-1-\frac{1}{p}}t^{-1}\right\},\ j\geq 2,\end{cases}$$

and

$$\begin{cases} \left|\left(w_1(t,x)-w_{1+}\right)w_{3x}(t,x)\right|\leq O(1)\delta_1\delta_3\varepsilon\exp\left(-2d_1\varepsilon t\right),\\ \left|\left(w_3(t,x)-w_{3-}\right)w_{1x}(t,x)\right|\leq O(1)\delta_1\delta_3\varepsilon\exp\left(-2d_1\varepsilon t\right).\end{cases}$$

Especially, there exists a constant $C(p)>0$ such that for $p>1$

$$\begin{cases} \left\|\left(w_1(t,x)-w_{1+}\right)w_{3x}(t,x)\right\|_{L^p}\leq C(p)\left(\delta_1\delta_3\varepsilon\right)^{1-\frac{1}{p}}\exp\left(-2d_1\left(1-\frac{1}{p}\right)\varepsilon t\right),\\ \left\|\left(w_3(t,x)-w_{3-}\right)w_{1x}(t,x)\right\|_{L^p}\leq C(p)\left(\delta_1\delta_3\varepsilon\right)^{1-\frac{1}{p}}\exp\left(-2d_1\left(1-\frac{1}{p}\right)\varepsilon t\right);\end{cases}$$

(iii) $\displaystyle\lim_{t\to+\infty}\sup_{x\in\mathbf{R}}\left|w_i(t,x)-w^{R_i}\left(\frac{x}{t}\right)\right|=0$.

Here $\delta_i=w_{i+}-w_{i-}$ is the wave strength of the i-th rarefaction wave.

From Lemma 3.1 and (1.23), we can easily deduce that $\left(\overline{\rho}(t,x),\overline{u}(t,x),\overline{\theta}(t,x)\right)$ is globally (w.r.t. t and x) well-defined and smooth. Moreover, from (1.20) and (1.23), $\left(\rho^{A_i},u^{A_i},\ \theta^{A_i}\right)(t,x)(i=1,3)$ satisfies

$$\begin{cases}\rho_t^{A_i}+\left(\rho^{A_i}u_1^{A_i}\right)_x=0,\\ u_{1t}^{A_i}+u_1^{A_i}u_{1x}^{A_i}+\frac{2}{3}\theta_x^{A_i}+\frac{2\theta^{A_i}}{3\rho^{A_i}}\rho_x^{A_i}=0,\\ \theta_t^{A_i}+u_1^{A_i}\theta_x^{A_i}+\frac{2}{3}\theta^{A_i}u_{1x}^{A_i}=0.\end{cases}\tag{3.3}$$

Therefore, we have from (1.22) and (1.23) that

$$\begin{cases}\overline{\rho}_x(t,x)=-\frac{3}{\sqrt{15k}}\overline{\rho}^{\frac{2}{3}}(t,x)\exp\left(-\frac{\overline{S}}{2}\right)u_{1x}^{A_1}(t+t_0,x)\\ \qquad\qquad+\frac{3}{\sqrt{15k}}\overline{\rho}^{\frac{2}{3}}(t,x)\exp\left(-\frac{\overline{S}}{2}\right)u_{1x}^{A_3}(t+t_0,x)+E_1(t,x),\\ \overline{\rho}_t(t,x)=\left\{\frac{3}{\sqrt{15k}}\overline{\rho}^{\frac{2}{3}}(t,x)\overline{u}_1(t,x)\exp\left(-\frac{\overline{S}}{2}\right)-\overline{\rho}(t,x)\right\}u_{1x}^{A_1}(t+t_0,x)\\ \qquad\qquad-\left\{\frac{3}{\sqrt{15k}}\overline{\rho}^{\frac{2}{3}}(t,x)\overline{u}_1(t,x)\exp\left(-\frac{\overline{S}}{2}\right)+\overline{\rho}(t,x)\right\}u_{1x}^{A_3}(t+t_0,x)+E_2(t,x),\\ \overline{u}_{1t}(t,x)=\left\{\frac{\sqrt{15k}}{3}\overline{\rho}^{\frac{1}{3}}(t,x)\exp\left(\frac{\overline{S}}{2}\right)-\overline{u}_1(t,x)\right\}u_{1x}^{A_1}(t+t_0,x)\\ \qquad\qquad-\left\{\frac{\sqrt{15k}}{3}\overline{\rho}^{\frac{1}{3}}(t,x)\exp\left(\frac{\overline{S}}{2}\right)+\overline{u}_1(t,x)\right\}u_{1x}^{A_3}(t+t_0,x)+E_3(t,x),\end{cases}\tag{3.4}$$

where

$$\begin{cases}E_1(t,x)=-\left\{\frac{3}{\sqrt{15k}}\left(\rho^{A_1}(t+t_0,x)\right)^{\frac{2}{3}}-\frac{3}{\sqrt{15k}}\overline{\rho}^{\frac{2}{3}}(t,x)\right\}\exp\left(-\frac{\overline{S}}{2}\right)u_{1x}^{A_1}(t+t_0,x)\\ \qquad\qquad+\frac{3\exp\left(-\frac{\overline{S}}{2}\right)}{\sqrt{15k}}\left\{\left(\rho^{A_3}(t+t_0,x)\right)^{\frac{2}{3}}-\overline{\rho}^{\frac{2}{3}}(t,x)\right\}u_{1x}^{A_3}(t+t_0,x),\\ E_2(t,x)=\Bigg\{\left[\frac{3}{\sqrt{15k}}\left(\rho^{A_1}(t+t_0,x)\right)^{\frac{2}{3}}u_1^{A_1}(t+t_0,x)\exp\left(-\frac{\overline{S}}{2}\right)-\rho^{A_1}(t+t_0,x)\right]\\ \qquad\qquad-\left[\frac{3}{\sqrt{15k}}\overline{\rho}^{\frac{2}{3}}(t,x)\overline{u}_1(t,x)\exp\left(-\frac{\overline{S}}{2}\right)-\overline{\rho}(t,x)\right]\Bigg\}u_{1x}^{A_1}(t+t_0,x)\\ \qquad\qquad-\Bigg\{\left[\frac{3}{\sqrt{15k}}\left(\rho^{A_3}(t+t_0,x)\right)^{\frac{2}{3}}u_1^{A_3}(t+t_0,x)\exp\left(-\frac{\overline{S}}{2}\right)+\rho^{A_3}(t+t_0,x)\right]\\ \qquad\qquad-\left[\frac{3}{\sqrt{15k}}\overline{\rho}^{\frac{2}{3}}(t,x)\overline{u}_1(t,x)\exp\left(-\frac{\overline{S}}{2}\right)+\overline{\rho}(t,x)\right]\Bigg\}u_{1x}^{A_3}(t+t_0,x),\\ E_3(t,x)=-\Bigg\{\left[u_1^{A_1}(t+t_0,x)-\frac{\sqrt{15k}}{3}\left(\rho^{A_1}(t+t_0,x)\right)^{\frac{1}{3}}\exp\left(\frac{\overline{S}}{2}\right)\right]\\ \qquad\qquad-\left[\overline{u}_1(t,x)-\frac{\sqrt{15k}}{3}\overline{\rho}^{\frac{1}{3}}(t,x)\exp\left(\frac{\overline{S}}{2}\right)\right]\Bigg\}u_{1x}^{A_1}(t+t_0,x)\\ \qquad\qquad-\Bigg\{\left[u_1^{A_3}(t+t_0,x)+\frac{\sqrt{15k}}{3}\left(\rho^{A_3}(t+t_0,x)\right)^{\frac{1}{3}}\exp\left(\frac{\overline{S}}{2}\right)\right]\\ \qquad\qquad-\left[\overline{u}_1(t,x)+\frac{\sqrt{15k}}{3}\overline{\rho}^{\frac{2}{3}}(t,x)\exp\left(\frac{\overline{S}}{2}\right)\right]\Bigg\}u_{1x}^{A_3}(t+t_0,x).\end{cases}\tag{3.5}$$

Furthermore, $\left(\overline{\rho}(t,x),\overline{u}(t,x),\overline{\theta}(t,x)\right)$ has the following decay properties.

Lemma 3.2 *The approximate rarefaction wave $(\overline{\rho}, \overline{u}, \overline{\theta})(t,x)$ constructed in (1.22) has the following properties*

(i) $u^{A_i}_{1x}(t,x) > 0,\ \forall (t,x) \in \mathbf{R}_+ \times \mathbf{R},\ i = 1,3$;

(ii) For any $p (1 \leq p \leq \infty)$*, there exists a constant* $C(p) > 0$*, depending only on* p*, such that*

$$\begin{cases} \left\| \left(\overline{\rho}, \overline{u}, \overline{\theta}\right)_x (t,x) \right\|_{L^p} \leq C(p) \min\left\{ \varepsilon^{1-\frac{1}{p}},\ (t+t_0)^{-1+\frac{1}{p}} \right\}, \\ \left\| \frac{\partial^j}{\partial x^j} \left(\overline{\rho}, \overline{u}, \overline{\theta}\right)(t,x) \right\|_{L^p} \leq C(p) \min\left\{ \varepsilon^{j-\frac{1}{p}},\ \varepsilon^{j-1-\frac{1}{p}} t^{-1} \right\},\ j \geq 2, \end{cases}$$

and

$$\begin{cases} \left| \left(\overline{\rho}_x, \overline{\theta}_x\right)(t,x) \right| \leq O(1) \left(\overline{u}_{1x}(t,x) + |E_1|\right), \\ \left\| (E_1, E_2, E_3)(t,x) \right\|_{L^p} \leq O(1) \varepsilon^{1-\frac{1}{p}} \exp\left(-2 \left(\frac{1}{\varepsilon} + d_1 \varepsilon t \right) \left(1 - \frac{1}{p} \right) \right); \end{cases} \tag{3.6}$$

(iii) $\lim\limits_{t \to +\infty} \sup\limits_{x \in \mathbf{R}} \left| \left(\overline{\rho}, \overline{u}, \overline{\theta}\right)(t,x) - \left(\rho^R, u^R, \theta^R\right)\left(\frac{x}{t}\right) \right| = 0.$

Precisely, Lemma 3.2 implies that for $\varepsilon > 0$ sufficiently small

$$\begin{cases} \int_0^t \left(\|(\overline{u}_{1x}, E_1)\|_{L^2}^{\frac{5}{2}} + \|\overline{u}_{1x}\|_{L^3}^3 + \|E_1\|_{L^2}^2 + \left\|(\overline{u}_{1xx}, \overline{\theta}_{xx})\right\|_{L^1}^{\frac{5}{4}} \right)(\tau) d\tau \leq O(1) \varepsilon^{\frac{1}{8}}, \\ \int_0^t \left(\|(\overline{u}_{1x}, E_1)\|_{L^2}^3 + \left\|(\overline{u}_{1xx}, \overline{\theta}_{xx})\right\|_{L^1}^{\frac{3}{2}} \right)(\tau) \left\| \sqrt{\eta(\tau)} \right\|_{L^2}^2 d\tau \\ \qquad \leq O(1) \int_0^t \left((1+\tau)^{-\frac{3}{2}} + \varepsilon^{\frac{3}{2}} \exp(-3 d_1 \varepsilon \tau) \right) \left\| \sqrt{\eta(\tau)} \right\|_{L^2}^2 d\tau, \end{cases} \tag{3.7}$$

where η is the entropy defined in (2.11).

All these will be used later in energy estimates.

4 Energy Estimates

In this section, we will give a proof of Theorem 1.1 by energy method. Denote ∂^α the differential operator $\partial^\alpha = \partial^{(\alpha_0, \alpha_1)} = \partial_t^{\alpha_0} \partial_x^{\alpha_1}$, $|\alpha| = \alpha_0 + \alpha_1$, where α_0 and α_1 are nonnegative integers. Set

$$\begin{cases} \tilde{\rho}(t,x) = \rho(t,x) - \overline{\rho}(t,x), \\ \tilde{u}(t,x) = u(t,x) - \overline{u}(t,x), \\ \tilde{\theta}(t,x) = \theta(t,x) - \overline{\theta}(t,x), \\ \tilde{\mathbf{G}}(t,x,\xi) = \mathbf{G}(t,x,\xi) - \overline{\mathbf{G}}(t,x,\xi) \end{cases}$$

with

$$\overline{\mathbf{G}}(t,x,\xi) = \frac{1}{R\theta(t,x)} L^{-1}_{\mathbf{M}_{[(\rho,u,\theta)(t,x)]}} \left\{ \mathbf{P}_1 \left[\xi_1 \left(\frac{|\xi - u(t,x)|^2}{2\theta(t,x)} \overline{\theta}_x(t,x) + \xi_1 \cdot \overline{u}_{1x} \right) \mathbf{M}(t,x) \right] \right\}. \tag{4.1}$$

Here we subtract $\overline{\mathbf{G}}(t,x,\xi)$ from $\mathbf{G}(t,x,\xi)$ because $\left\| \left(\overline{u}_x, \overline{\theta}_x\right)(t) \right\|_{L^2}^2$ is not integrable w.r.t t. Since the local existence of solutions is well-known, all we need is to close the following *a priori* assumption

$$N(t)^2 = \sup_{0\le\tau\le t}\left\{\int_{\mathbf{R}}\eta(\tau)dx + \int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-} + \sum_{1\le|\alpha|\le 4}\frac{(\partial^\alpha\mathbf{M})^2+(\partial^\alpha\mathbf{G})^2}{\mathbf{M}_-} + \sum_{|\alpha|=5}\frac{(\partial^\alpha f)^2}{\mathbf{M}_-}\right)(\tau)d\xi dx\right\}$$
$$< \delta_0^2. \tag{4.2}$$

Here $\delta_0 > 0$ is a suitably chosen sufficiently small constant.

From (1.9), (4.2) yields the following $L^\infty_{(t,x)}$ estimates by Sobolev imbedding theorem.

$$\sup_{\tau\in[0,t],x\in\mathbf{R}}\left\{\left|(\tilde\rho,\tilde u,\tilde\theta)(\tau,x)\right| + \sum_{1\le|\alpha|\le 3}\left(|\partial^\alpha(\rho,u,\theta)(\tau,x)| + \left\|\frac{\partial^\alpha\mathbf{G}(\tau,x)}{\sqrt{\mathbf{M}_-(\tau,x)}}\right\|_{L^2_\xi}\right)\right\} < O(1)(\varepsilon+\delta_0), \tag{4.3}$$

where ε is the constant in the definition of the approximate rarefaction waves.

Under the *a priori* assumptions (4.2), by choosing δ_0 and ε to be sufficiently small, there exists a constant state $(\rho_-,u_-,\theta_-)(\rho_->0,\theta_->0)$ such that for all $(\tau,x)\in[0,t]\times\mathbf{R}$

$$\frac{1}{2}\theta(\tau,x) < \theta_- < \theta(\tau,x), \quad |\theta(\tau,x)-\theta_-| + |u(\tau,x)-u_-| + |\rho(\tau,x)-\rho_-| < \eta_0. \tag{4.4}$$

Thus, the microscopic $H-$theorem, i.e. (1.24) holds for the global Maxwellian $\mathbf{M}_- = \mathbf{M}_{[\rho_-,u_-,\theta_-]}$.

In the following three sub-sections, we will give the energy estimates on the entropy, derivatives with the weight of local Maxwellian $\mathbf{M}$, and the derivatives with the weight of global Maxwellian $\mathbf{M}_-$ respectively.

4.1 Lower order estimates

In this subsection, we will give the energy estimates on the entropy and the non-fluid component $\mathbf{G}(t,x,\xi)$.

From (2.8) and (2.11), integration over $[0,t)\times\mathbf{R}\times\mathbf{R}^3$ yields

$$\begin{aligned}\int_{\mathbf{R}}\eta(\tau)d\xi\Big|_{\tau=0}^{\tau=t} &= \int_0^t\int_{\mathbf{R}}\left[\nabla_{(\bar\rho,\bar u,\bar S)}\eta\cdot\left(\bar\rho,\bar u,\bar S\right)_t + \nabla_{(\bar\rho,\bar u,\bar S)}q\cdot\left(\bar\rho,\bar u,\bar S\right)_x\right]dxd\tau \\ &+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left[\xi_1\partial_x(\bar\theta\ln\mathbf{M}) - \tfrac{3}{2}\bar u_{1x}\xi_1^2\right]\mathbf{G}d\xi dxd\tau.\end{aligned} \tag{4.5}$$

From (1.11), we first estimates the last integral on the right hand side of (4.5).

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left[\xi_1\partial_x(\bar\theta\ln\mathbf{M}) - \tfrac{3}{2}\bar u_{1x}\xi_1^2\right]\mathbf{G}d\xi dxd\tau \\ =&\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left[\xi_1\partial_x(\bar\theta\ln\mathbf{M}) - \tfrac{3}{2}\bar u_{1x}\xi_1^2\right]L^{-1}_{\mathbf{M}}\Big(\mathbf{P}_1(\xi_1\partial_x\mathbf{M})\Big)d\xi dxd\tau \\ &+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left[\xi_1\partial_x(\bar\theta\ln\mathbf{M}) - \tfrac{3}{2}\bar u_{1x}\xi_1^2\right]L^{-1}_{\mathbf{M}}\Big(\mathbf{G}_t + \mathbf{P}_1(\xi_1\partial_x\mathbf{G}) - Q(\mathbf{G},\mathbf{G})\Big)d\xi dxd\tau \\ :=& I_1 + I_2.\end{aligned} \tag{4.6}$$

Here and in the sequel, we denote the corresponding terms in the summation by I_i^j without any ambiguity. Since

$$\begin{aligned}\mathbf{P}_1\left[\xi_1\partial_x(\bar\theta\ln\mathbf{M}) - \tfrac{3}{2}\bar u_{1x}\xi_1^2\right] &= -\tfrac{3}{2}\mathbf{P}_1\left[\xi_1\left(\tfrac{\bar\theta|\xi-u|^2}{2\theta}\right)_x + \xi_1^2\bar u_{1x}\right] \\ &= \tfrac{3}{2}\mathbf{P}_1\left[\xi_1\left(\xi\cdot\tilde u_x + \tfrac{\tilde\theta_x}{2\theta}|\xi-u|^2\right) + \tilde\theta\xi_1\left(\tfrac{|\xi-u|^2}{2\theta}\right)_x\right],\end{aligned}$$

we have

$$
\begin{aligned}
I_1 = & \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \tfrac{9}{4\theta} \left(\xi_1 \left(\xi \cdot \tilde{u}_x + \tilde{\theta}_x \tfrac{|\xi-u|^2}{2\theta}\right)\right) L_{\mathbf{M}}^{-1} \Big\{ \mathbf{P}_1 \left[\xi_1 \left(\xi \cdot \tilde{u}_x + \tilde{\theta}_x \tfrac{|\xi-u|^2}{2\theta}\right) \mathbf{M}\right] \Big\} d\xi dx d\tau \\
& + \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \tfrac{9}{4\theta} \left(\xi_1 \left(\xi \cdot \tilde{u} + \tilde{\theta} \tfrac{|\xi-u|^2}{2\theta}\right)_x\right) L_{\mathbf{M}}^{-1} \Big\{ \mathbf{P}_1 \left[\xi_1 \left(\xi_1 \cdot \overline{u}_{1x} + \overline{\theta}_x \tfrac{|\xi-u|^2}{2\theta}\right) \mathbf{M}\right] \Big\} d\xi dx d\tau \\
& + \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \tfrac{9}{4\theta} \tilde{\theta} \xi_1 \left(\tfrac{|\xi-u|^2}{2\theta}\right)_x L_{\mathbf{M}}^{-1} \Big\{ \mathbf{P}_1 \left[\xi_1 \left(\xi_1 \cdot \overline{u}_{1x} + \overline{\theta}_x \tfrac{|\xi-u|^2}{2\theta}\right) \mathbf{M}\right] \Big\} d\xi dx d\tau \\
:= & \sum_{j=1}^3 I_1^j.
\end{aligned}
\tag{4.7}
$$

Notice that

$$
\begin{cases}
\partial_x \Big\{ L_{\mathbf{M}}^{-1} h \Big\} = L_{\mathbf{M}}^{-1} h_x - 2 L_{\mathbf{M}}^{-1} \Big\{ Q\left(L_{\mathbf{M}}^{-1} h, \mathbf{M}_x\right) \Big\}, \\
L_{\mathbf{M}}^{-1} \Big\{ \mathbf{P}_1 \left[\xi_1 \left(\xi_1 \cdot \overline{u}_{1x} + \overline{\theta}_x \tfrac{|\xi-u|^2}{2\theta}\right) \mathbf{M}\right] \Big\} = L_{\mathbf{M}_{[1,u,\theta]}}^{-1} \Big\{ \mathbf{P}_1 \left[\xi_1 \left(\xi_1 \cdot \overline{u}_{1x} + \overline{\theta}_x \tfrac{|\xi-u|^2}{2\theta}\right) \mathbf{M}_{[1,u,\theta]}\right] \Big\}.
\end{cases}
$$

For some small constant $\lambda > 0$, we have from $(3.6)_1$, Lemma 2.1, and Corollary 2.1 that

$$
\begin{cases}
I_1^1 \leq -d_2 \int_0^t \int_{\mathbf{R}} \left(|\tilde{u}_x|^2 + |\tilde{\theta}_x|^2\right) dx d\tau, \\
I_1^3 \leq O(1) \int_0^t \int_{\mathbf{R}} |\tilde{\theta}| (|u_x| + |\theta_x|) \left(\left|\overline{\theta}_x\right| + |\overline{u}_{1x}|\right) dx d\tau \\
\quad \leq \lambda \int_0^t \int_{\mathbf{R}} \left(|\tilde{u}_x|^2 + |\tilde{\theta}_x|^2 + \overline{u}_{1x} \left|\tilde{\theta}\right|^2\right) dx d\tau + O(1) \int_0^t \int_{\mathbf{R}} \left(|\overline{u}_{1x}|^3 + E_1^2\right) dx d\tau,
\end{cases}
\tag{4.8}
$$

and

$$
\begin{aligned}
I_1^2 = & -\tfrac{9}{4} \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \xi_1 \left(\xi \cdot \tilde{u} + \tilde{\theta} \tfrac{|\xi-u|^2}{2\theta}\right) \partial_x \Big\{ L_{\mathbf{M}}^{-1} \Big\{ \tfrac{1}{\theta} \mathbf{P}_1 \left[\xi_1 \left(\xi_1 \cdot \overline{u}_{1x} + \overline{\theta}_x \tfrac{|\xi-u|^2}{2\theta}\right) \mathbf{M}\right] \Big\} \Big\} d\xi dx d\tau \\
\leq & O(1) \int_0^t \int_{\mathbf{R}} (|\tilde{\theta}| + |\tilde{u}|) \left(\left|\overline{\theta}_{xx}\right| + |\overline{u}_{1xx}| + (|\theta_x| + |u_x|)(|\overline{u}_{1x}| + |\overline{\theta}_x|)\right) dx d\tau \\
\leq & O(1) \int_0^t \int_{\mathbf{R}} (|\tilde{\theta}| + |\tilde{u}|) \left(\left|\overline{\theta}_{xx}\right| + |\overline{u}_{1xx}| + |\overline{u}_{1x}|^2 + E_1^2 + (|\tilde{\theta}_x| + |\tilde{u}_x|)^2\right) dx d\tau \\
\leq & \lambda \int_0^t \int_{\mathbf{R}} \left(|\tilde{u}_x|^2 + |\tilde{\theta}_x|^2\right) dx d\tau \\
& + O(1) \int_0^t \left(\|(\overline{u}_{1x}, E_1)\|_{L^2}^3 + \left\|(\overline{u}_{1xx}, \overline{\theta}_{xx})\right\|_{L^1}^{\frac{3}{2}}\right)(\tau) \left\|\sqrt{\eta(\tau)}\right\|_{L^2}^2 d\tau \\
& + O(1) \int_0^t \left(\|(\overline{u}_{1x}, E_1)(\tau)\|_{L^2}^{\frac{5}{2}} + \left\|(\overline{u}_{1xx}, \overline{\theta}_{xx})(\tau)\right\|_{L^1}^{\frac{5}{4}}\right) d\tau.
\end{aligned}
\tag{4.9}
$$

Similarly

$$
\begin{aligned}
I_2 \leq & \lambda \int_0^t \int_{\mathbf{R}} \left(|\tilde{u}_x|^2 + |\tilde{\theta}_x|^2 + \overline{u}_{1x} \left|\tilde{\theta}\right|^2\right) dx d\tau + O(1) \int_0^t \int_{\mathbf{R}} E_1^2 dx d\tau \\
& + O(1) \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \tfrac{\mathbf{G}_t^2 + (1+|\xi|)\mathbf{G}_x^2 + (1+|\xi|)^{-1} Q(\mathbf{G},\mathbf{G})^2}{\mathbf{M}} d\xi dx d\tau.
\end{aligned}
\tag{4.10}
$$

Substituting (4.6)-(4.10) into (4.5), we have

$$
\begin{aligned}
&\int_{\mathbf{R}}\eta(\tau)d\xi\Big|_{\tau=0}^{\tau=t}+\int_0^t\int_{\mathbf{R}}\left(|\tilde{u}_x|^2+|\tilde{\theta}_x|^2\right)dxd\tau\\
\le&\int_0^t\int_{\mathbf{R}}\left[\nabla_{(\overline{\rho},\overline{u},\overline{S})}\eta\cdot\left(\overline{\rho},\overline{u},\overline{S}\right)_t+\nabla_{(\overline{\rho},\overline{u},\overline{S})}q\cdot\left(\overline{\rho},\overline{u},\overline{S}\right)_x\right]dxd\tau\\
&+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\mathbf{G}_t^2+(1+|\xi|)\mathbf{G}_x^2+(1+|\xi|)^{-1}Q(\mathbf{G},\mathbf{G})^2}{\mathbf{M}}d\xi dxd\tau\\
&+\lambda\int_0^t\int_{\mathbf{R}}\overline{u}_{1x}|\tilde{\theta}|^2dxd\tau+O(1)\int_0^t\left(\|(\overline{u}_{1x},E_1)(\tau)\|_{L^2}^3+\left\|(\overline{u}_{1xx},\overline{\theta}_{xx})(\tau)\right\|_{L^1}^{\frac{3}{2}}\right)\left\|\sqrt{\eta(\tau)}\right\|_{L^2}^2d\tau\\
&+O(1)\int_0^t\left(\|(\overline{u}_{1x},E_1)(\tau)\|_{L^2}^{\frac{5}{2}}+\|\overline{u}_{1x}(\tau)\|_{L^3}^3+\left\|(\overline{u}_{1xx},\overline{\theta}_{xx})(\tau)\right\|_{L^1}^{\frac{5}{4}}+\|E_1(\tau)\|_{L^2}^2\right)d\tau.
\end{aligned}
\tag{4.11}
$$

Now we estimate the first term on the right hand side of (4.11) where the sign of $\overline{u}_{1x}$ is used. Direct calculations gives

$$
\begin{aligned}
&\nabla_{(\overline{\rho},\overline{u},\overline{S})}\eta\cdot\left(\overline{\rho},\overline{u},\overline{S}\right)_t+\nabla_{(\overline{\rho},\overline{u},\overline{S})}q\cdot\left(\overline{\rho},\overline{u},\overline{S}\right)_x\\
=&\,\eta_{\overline{\rho}}E_2+\eta_{\overline{u}_1}E_3+q_{\overline{\rho}}E_1-H_1\left(\rho,u,\theta;\overline{\rho},\overline{u},\overline{\theta}\right)u_{1x}^{A_1}-H_2\left(\rho,u,\theta;\overline{\rho},\overline{u},\overline{\theta}\right)u_{1x}^{A_3}.
\end{aligned}
\tag{4.12}
$$

Here

$$
\begin{cases}
H_1\left(\rho,u,\theta;\overline{\rho},\overline{u},\overline{\theta}\right)=\frac{3}{2}\rho\left(u_1-\overline{u}_1\right)^2+\frac{3}{2}k\left(\rho^{\frac{5}{3}}\exp(S)-\overline{\rho}^{\frac{5}{3}}\exp\left(\overline{S}\right)\right)\\
\qquad-\frac{5}{2}k\overline{\rho}^{\frac{2}{3}}\exp\left(\overline{S}\right)\left(\rho-\overline{\rho}\right)-\frac{3}{2}k\overline{\rho}^{\frac{2}{3}}\exp\left(\overline{S}\right)\rho\left(S-\overline{S}\right)\\
\qquad-\frac{3\sqrt{15k}}{10}\overline{\rho}^{\frac{1}{3}}\exp\left(\frac{\overline{S}}{2}\right)\rho\left(u_1-\overline{u}_1\right)\left(S-\overline{S}\right),\\
H_2\left(\rho,u,\theta;\overline{\rho},\overline{u},\overline{\theta}\right)=\frac{3}{2}\rho\left(u_1-\overline{u}_1\right)^2+\frac{3}{2}k\left(\rho^{\frac{5}{3}}\exp(S)-\overline{\rho}^{\frac{5}{3}}\exp\left(\overline{S}\right)\right)\\
\qquad-\frac{5}{2}k\overline{\rho}^{\frac{2}{3}}\exp\left(\overline{S}\right)\left(\rho-\overline{\rho}\right)-\frac{3}{2}k\overline{\rho}^{\frac{2}{3}}\exp\left(\overline{S}\right)\rho\left(S-\overline{S}\right)\\
\qquad+\frac{3\sqrt{15k}}{10}\overline{\rho}^{\frac{1}{3}}\exp\left(\frac{\overline{S}}{2}\right)\rho\left(u_1-\overline{u}_1\right)\left(S-\overline{S}\right).
\end{cases}
\tag{4.13}
$$

From (2.12) and (4.13), we can get that

$$
\begin{cases}
\eta_{\overline{\rho}}=-\frac{3}{2}k\overline{\rho}^{-\frac{1}{3}}\exp\left(\overline{S}\right)\rho\left(S-\overline{S}\right)-\frac{5}{2}k\exp\left(\overline{S}\right)\overline{\rho}^{-\frac{1}{3}}\left(\rho-\overline{\rho}\right),\\
\eta_{\overline{u}_1}=-\frac{3}{2}\rho\left(u_1-\overline{u}_1\right),\\
q_{\overline{\rho}}=-\frac{3}{2}k\overline{\rho}^{-\frac{1}{3}}\exp\left(\overline{S}\right)\rho u_1\left(S-\overline{S}\right)-\frac{5}{2}k\exp\left(\overline{S}\right)\overline{\rho}^{-\frac{1}{3}}\left(\rho u_1-\overline{\rho}\overline{u}_1\right),
\end{cases}
\tag{4.14}
$$

$$
\begin{cases}
\nabla_{(\rho,u_1,S)}H_i\left(\overline{\rho},\overline{u},\overline{\theta};\overline{\rho},\overline{u},\overline{\theta}\right)=(0,0,0),\\
\nabla^2_{(\rho,u_1,S)}H_i\left(\overline{\rho},\overline{u},\overline{\theta};\overline{\rho},\overline{u},\overline{\theta}\right)\\
\quad=\begin{pmatrix}
\frac{5}{3}k\overline{\rho}^{-\frac{1}{3}}\exp\left(\overline{S}\right) & 0 & k\overline{\rho}^{\frac{2}{3}}\exp\left(\overline{S}\right)\\
0 & 3\overline{\rho} & (-1)^i\frac{3\sqrt{15k}}{10}\overline{\rho}^{\frac{4}{3}}\exp\left(\frac{\overline{S}}{2}\right)\\
k\overline{\rho}^{\frac{2}{3}}\exp\left(\overline{S}\right) & (-1)^i\frac{3\sqrt{15k}}{10}\overline{\rho}^{\frac{4}{3}}\exp\left(\frac{\overline{S}}{2}\right) & \frac{3}{2}k\overline{\rho}^{\frac{5}{3}}\exp\left(\overline{S}\right)
\end{pmatrix}.
\end{cases}
\tag{4.15}
$$

Since $\nabla^2_{(\rho,u_1,S)}H_i$, the Hessian of H_i, are positive definite at $(\rho,u,\theta)=\left(\overline{\rho},\overline{u},\overline{\theta}\right)$. Thus around the approximate rarefaction waves profile $\left(\overline{\rho},\overline{u},\overline{\theta}\right)$, there exists a positive constant d_3 such that

$$-H_1\left(\rho,u,\theta;\overline{\rho},\overline{u},\overline{\theta}\right)u_{1x}^{A_1}-H_2\left(\rho,u,\theta;\overline{\rho},\overline{u},\overline{\theta}\right)u_{1x}^{A_3}\le -d_3\overline{u}_{1x}\left(|\tilde{\rho}|^2+|\tilde{u}|^2+|\tilde{\theta}|^2\right). \quad (4.16)$$

Furthermore, (2.13), (4.14), and Lemma 3.2 implies that

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}}\left(\eta_{\overline{\rho}}E_2+\eta_{\overline{u}_1}E_3+q_{\overline{\rho}}E_1\right)dxd\tau\le O(1)\int_0^t\left\|\sqrt{\eta(\tau)}\right\|_{L^2}\|(E_1,E_2,E_3)(\tau)\|_{L^2}d\tau\\&\quad\le O(1)\left(\int_0^t\|(E_1,E_2,E_3)(\tau)\|_{L^2}d\tau+\int_0^t\left\|\sqrt{\eta(\tau)}\right\|^2_{L^2}\|(E_1,E_2,E_3)(\tau)\|_{L^2}d\tau\right)\\&\quad\le O(1)\varepsilon^{-\frac12}\exp\left(-\tfrac{1}{\varepsilon}\right)+O(1)\sqrt{\varepsilon}\exp\left(-\tfrac{1}{\varepsilon}\right)\int_0^t\exp(-\varepsilon d_1\tau)\left\|\sqrt{\eta(\tau)}\right\|^2_{L^2}d\tau.\end{aligned} \quad (4.17)$$

Plugging (3.7), (4.16), and (4.17) into (4.11), we have from Gronwall's inequality the following entropy estimate

$$\begin{aligned}&\int_{\mathbf{R}}\eta(t)d\xi+\int_0^t\int_{\mathbf{R}}\left(|\tilde{u}_x|^2+|\tilde{\theta}_x|^2+\overline{u}_{1x}(|\tilde{\rho}|^2+|\tilde{u}|^2+|\tilde{\theta}|^2)\right)dxd\tau\\&\le O(1)\left(\int_{\mathbf{R}}\eta(0)dx+\varepsilon^{\frac18}\right)+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{\mathbf{G}_t^2+(1+|\xi|)\mathbf{G}_x^2+(1+|\xi|)^{-1}Q(\mathbf{G},\mathbf{G})^2}{\mathbf{M}}d\xi dxd\tau.\end{aligned} \quad (4.18)$$

Next we will derive the lower order estimates for the microscopic part $\tilde{\mathbf{G}}$ which solves

$$\tilde{\mathbf{G}}_t-L_{\mathbf{M}}\tilde{\mathbf{G}}=-\frac{1}{R\theta}\mathbf{P}_1\left[\xi_1\left(\frac{|\xi-u|^2}{2\theta}\tilde{\theta}_x+\xi\cdot\tilde{u}_x\right)\mathbf{M}\right]-\mathbf{P}_1(\xi_1\mathbf{G}_x)+Q(\mathbf{G},\mathbf{G})-\overline{\mathbf{G}}_t. \quad (4.19)$$

Multiplying (4.19) by $\frac{\tilde{\mathbf{G}}}{\mathbf{M}}$ and integrating the product over $[0,t]\times\mathbf{R}\times\mathbf{R}^3$, we have from (1.14) that

$$\begin{aligned}&\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi dx\Big|_{\tau=0}^{\tau=t}+\sigma\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi dxd\tau\le-\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{\tilde{\mathbf{G}}^2}{\mathbf{M}^2}\mathbf{M}_td\xi dxd\tau\\&-\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{\tilde{\mathbf{G}}}{\mathbf{M}}\left\{\tfrac{1}{R\theta}\mathbf{P}_1\left[\xi_1\left(\tfrac{|\xi-u|^2}{2\theta}\tilde{\theta}_x+\xi\cdot\tilde{u}_x\right)\mathbf{M}\right]+\mathbf{P}_1(\xi_1\mathbf{G}_x)-Q(\mathbf{G},\mathbf{G})+\overline{\mathbf{G}}_t\right\}d\xi dxd\tau\\&:=I_3+I_4.\end{aligned} \quad (4.20)$$

From (4.1), (4.2), (4.3), Lemma 2.1, and Corollary 2.1, we have

$$I_3\le O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-}d\xi dxd\tau, \quad (4.21)$$

$$\begin{aligned}I_4\le&\ \tfrac{\sigma}{2}\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi dxd\tau\\&+O(1)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)^{-1}}{\mathbf{M}}\Bigg\{\left|\mathbf{P}_1\left[\xi_1\left(\tfrac{|\xi-u|^2}{2\theta}\tilde{\theta}_x+\xi\cdot\tilde{u}_x\right)\mathbf{M}\right]\right|^2\\&+|\mathbf{P}_1(\xi_1\mathbf{G}_x)|^2+|Q(\mathbf{G},\mathbf{G})|^2+\left|\overline{\mathbf{G}}_t\right|^2\Bigg\}d\xi dxd\tau\\\le&\ \tfrac{\sigma}{2}\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi dxd\tau+O(1)\int_0^t\int_{\mathbf{R}}\left|(\tilde{u}_x,\tilde{\theta}_x)\right|^2dxd\tau\\&+O(1)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)\mathbf{G}_x^2+(1+|\xi|)^{-1}Q(\mathbf{G},\mathbf{G})^2+(1+|\xi|)^{-1}|\overline{\mathbf{G}}_t|^2}{\mathbf{M}}d\xi dxd\tau,\end{aligned} \quad (4.22)$$

and

$$
\begin{aligned}
&\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)^{-1}|\overline{\mathbf{G}}_t|^2}{\mathbf{M}}d\xi dxd\tau\\
&\leq\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)^{-1}}{\mathbf{M}}\left|L_{\mathbf{M}}^{-1}\left\{\mathbf{P}_1\left[\tfrac{\xi_1}{R\theta}\left(\tfrac{|\xi-u|^2}{2\theta}\overline{\theta}_x+\xi_1\cdot\overline{u}_{1x}\right)\mathbf{M}\right]_t\right\}\right|^2d\xi dxd\tau\\
&\quad+4\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)^{-1}}{\mathbf{M}}\left|Q\left(\mathbf{P}_1\left[\tfrac{\xi_1}{R\theta}\left(\tfrac{|\xi-u|^2}{2\theta}\overline{\theta}_x+\xi_1\cdot\overline{u}_{1x}\right)\mathbf{M}\right],\mathbf{M}_t\right)\right|^2d\xi dxd\tau\\
&\leq O(1)\left(\varepsilon^{\frac18}+(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\left|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)\right|^2dxd\tau\right).
\end{aligned}\tag{4.23}
$$

Putting (4.21)-(4.23) into (4.20) yields

$$
\begin{aligned}
&\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi dx\Big|_{\tau=0}^{\tau=t}+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi dxd\tau\\
&\leq O(1)\varepsilon^{\frac18}+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\quad+O(1)\int_0^t\int_{\mathbf{R}}\left((\varepsilon+\delta_0)\left|\tilde\rho_x\right|^2+\left|(\tilde u_x,\tilde\theta_x)\right|^2\right)dxd\tau\\
&\quad+O(1)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)\mathbf{G}_x^2+(1+|\xi|)^{-1}Q(\mathbf{G},\mathbf{G})^2}{\mathbf{M}}d\xi dxd\tau.
\end{aligned}\tag{4.24}
$$

Similarly, using the weight $\mathbf{M}_-$ instead of $\mathbf{M}$, we have

$$
\begin{aligned}
&\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-}d\xi dx\Big|_{\tau=0}^{\tau=t}+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}_-}d\xi dxd\tau\\
&\leq O(1)\varepsilon^{\frac18}+O(1)\int_0^t\int_{\mathbf{R}}\left((\varepsilon+\delta_0)\left|\tilde\rho_x\right|^2+\left|(\tilde u_x,\tilde\theta_x)\right|^2\right)dxd\tau\\
&\quad+O(1)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)\mathbf{G}_x^2+(1+|\xi|)^{-1}Q(\mathbf{G},\mathbf{G})^2}{\mathbf{M}_-}d\xi dxd\tau.
\end{aligned}\tag{4.25}
$$

Since for $\mathbf{M}_i=\mathbf{M}_-$ or $\mathbf{M}$, Lemma 2.1 and (4.3) give

$$
\begin{aligned}
&\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{((1+|\xi|)^{-1}Q(\mathbf{G},\mathbf{G})^2}{\mathbf{M}_i}d\xi dxd\tau\leq O(1)\int_0^t\int_{\mathbf{R}}\left(\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)\mathbf{G}^2}{\mathbf{M}_i}\right)\cdot\left(\int_{\mathbf{R}^3}\tfrac{\mathbf{G}^2}{\mathbf{M}_i}\right)dxd\tau\\
&\leq O(1)\int_0^t\int_{\mathbf{R}}\left(\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)(\tilde{\mathbf{G}}^2+\overline{\mathbf{G}}^2)}{\mathbf{M}_i}\right)\cdot\left(\int_{\mathbf{R}^3}\tfrac{\tilde{\mathbf{G}}^2+\overline{\mathbf{G}}^2}{\mathbf{M}_i}\right)dxd\tau\\
&\leq O(1)\varepsilon^{\frac18}+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)\tilde{\mathbf{G}}_x^2}{\mathbf{M}_i}d\xi dxd\tau,
\end{aligned}\tag{4.26}
$$

we have from (4.18), (4.24), and (4.25) the following two estimates on the entropy η and $\tilde{\mathbf{G}}$.

$$
\begin{aligned}
&\int_{\mathbf{R}}\eta(t)d\xi+\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi d\tau\Big|_{t=0}^{\tau=t}+\int_0^t\int_{\mathbf{R}}\left(|(\tilde u_x,\tilde\theta_x)|^2+\overline{u}_{1x}|(\tilde\rho,\tilde u,\tilde\theta)|^2\right)dxd\tau\\
&\quad+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi dxd\tau\leq O(1)\left(\varepsilon^{\frac18}+\int_{\mathbf{R}}\eta(0)dx\right)\\
&\quad+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left|\tilde\rho_x\right|^2dxd\tau\\
&\quad+O(1)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left((\varepsilon+\delta_0)\tfrac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-}+\tfrac{(1+|\xi|)\left(\mathbf{G}_t^2+\mathbf{G}_x^2\right)}{\mathbf{M}}\right)d\xi dxd\tau,
\end{aligned}\tag{4.27}
$$

and

$$
\begin{aligned}
&\int_{\mathbf{R}}\eta(t)d\xi+\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-}d\xi d\tau\Big|_{t=0}^{\tau=t}+\int_0^t\int_{\mathbf{R}}\Big(|(\tilde{u}_x,\tilde{\theta}_x)|^2+\overline{u}_{1x}|(\tilde{\rho},\tilde{u},\tilde{\theta})|^2\Big)dxd\tau\\
&\quad+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}_-}d\xi dxd\tau\leq O(1)\left(\varepsilon^{\frac{1}{8}}+\int_{\mathbf{R}}\eta(0)dx\right)\\
&\quad+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}|\tilde{\rho}_x|^2\,dxd\tau\\
&\quad+O(1)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\big(\mathbf{G}_t^2+\mathbf{G}_x^2\big)}{\mathbf{M}_-}d\xi dxd\tau.
\end{aligned}
\tag{4.28}
$$

Notice that in the above two estimates (4.27) and (4.28), the double integral of $\tilde{\rho}_x^2$ and $\tilde{\rho}_t^2$ are missing. In the following, we will show that they can be recovered from the conservation laws. From (1.9), (1.19), and (3.3), since $\left(\tilde{\rho},\tilde{u},\tilde{\theta}\right)(t,x)$ solves

$$
\begin{cases}
\tilde{\rho}_t+\left(\rho\tilde{u}_1\right)_x=-E_4,\\
\tilde{u}_{1t}+\tilde{u}_1\tilde{u}_{1x}+\frac{2}{3}\tilde{\theta}_x+\frac{2\theta}{3\rho}\tilde{\rho}_x=-\int_{\mathbf{R}^3}\frac{\xi_1^2\mathbf{G}_x}{\rho}d\xi-E_5,\\
\tilde{u}_{2t}+\tilde{u}_1\tilde{u}_{2x}=-\int_{\mathbf{R}^3}\frac{\xi_1\xi_2\mathbf{G}_x}{\rho}d\xi-\overline{u}_1\tilde{u}_{2x},\\
\tilde{u}_{3t}+\tilde{u}_1\tilde{u}_{3x}=-\int_{\mathbf{R}^3}\frac{\xi_1\xi_3\mathbf{G}_x}{\rho}d\xi-\overline{u}_1\tilde{u}_{3x},\\
\tilde{\theta}_t+\frac{2}{3}\tilde{\theta}\tilde{u}_{1x}+\tilde{u}_1\tilde{\theta}_x=\int_{\mathbf{R}^3}\frac{\xi_1\left(\xi\cdot u-\frac{1}{2}|\xi|^2\right)\mathbf{G}_x}{\rho}d\xi-E_6.
\end{cases}
\tag{4.29}
$$

where

$$
\begin{cases}
E_4=\left(\overline{\rho}\tilde{u}_1+\overline{u}_1\tilde{\rho}\right)_x+\rho_x^{A_1}\left(u_1^{A_3}-u_{1m}\right)+\rho_x^{A_3}\left(u_1^{A_1}-u_{1m}\right)\\
\qquad+u_{1x}^{A_1}\left(\rho^{A_3}-\rho_m\right)+u_{1x}^{A_3}\left(\rho^{A_1}-\rho_m\right),\\
E_5=\tilde{u}_1\overline{u}_{1x}+\overline{u}_1\tilde{u}_{1x}+u_{1x}^{A_1}\left(u_1^{A_3}-u_{1m}\right)+u_{1x}^{A_3}\left(u_1^{A_1}-u_{1m}\right)\\
\qquad+\frac{2}{3}\left\{\frac{\overline{\rho}\tilde{\theta}-\overline{\theta}\tilde{\rho}}{\rho\overline{\rho}}\overline{\rho}_x+\frac{\rho^{A_1}\left(\theta^{A_3}-\theta_m\right)-\theta^{A_1}\left(\rho^{A_3}-\rho_m\right)}{\overline{\rho}\rho^{A_1}}\rho_x^{A_1}+\frac{\rho^{A_3}\left(\theta^{A_1}-\theta_m\right)-\theta^{A_3}\left(\rho^{A_1}-\rho_m\right)}{\overline{\rho}\rho^{A_3}}\rho_x^{A_3}\right\},\\
E_6=\frac{2}{3}\left\{\left(\tilde{\theta}\overline{u}_{1x}+\overline{\theta}\tilde{u}_{1x}\right)+u_{1x}^{A_1}\left(\theta^{A_3}-\theta_m\right)+u_{1x}^{A_3}\left(\theta^{A_1}-\theta_m\right)\right\}\\
\qquad+\left\{\left(\tilde{u}_1\overline{\theta}_x+\overline{u}_1\tilde{\theta}_x\right)+\theta_x^{A_1}\left(u_1^{A_3}-u_{1m}\right)+\theta_x^{A_3}\left(u_1^{A_1}-u_{1m}\right)\right\},
\end{cases}
\tag{4.30}
$$

we have by Lemma 3.2 that

$$
\begin{aligned}
\int_0^t\int_{\mathbf{R}}|\tilde{\rho}_x|^2dxd\tau\leq\;&O(1)\varepsilon^{\frac{1}{8}}+O(1)\int_{\mathbf{R}}|(\tilde{u},\tilde{\rho}_x)|^2(t)dx+O(1)\int_{\mathbf{R}}|(\tilde{u},\tilde{\rho}_x)|^2(0)dx\\
&+O(1)\int_0^t\int_{\mathbf{R}}\left(|\tilde{\theta}_x|^2+|\tilde{u}_x|^2+\int_{\mathbf{R}^3}\frac{\mathbf{G}_x^2}{\mathbf{M}}d\xi\right)dxd\tau\\
&+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\overline{u}_{1x}\left(|\tilde{u}|^2+|\tilde{\theta}|^2+|\tilde{\rho}|^2\right)dxd\tau
\end{aligned}
\tag{4.31}
$$

and

$$
\int_0^t\int_{\mathbf{R}}|\tilde{\rho}_t|^2\,dxd\tau\leq O(1)\varepsilon^{-\frac{1}{2}}\exp\left(-\frac{1}{\varepsilon}\right)+O(1)\int_0^t\int_{\mathbf{R}}\left(|(\tilde{\rho}_x,\tilde{u}_x)|^2+\overline{u}_{1x}|(\tilde{\rho},\tilde{u})|^2\right)dxd\tau.
\tag{4.32}
$$

Hence, (4.27)-(4.28) and (4.31)-(4.32) give the complete lower order estimates.

4.2 Higher order estimates w.r.t. M

In this subsection, we will consider higher order energy estimates of $\partial^\alpha \mathbf{M}, \partial^\alpha \mathbf{G}$, and $\partial^\beta f$ for $1 \leq |\alpha| \leq 4, 2 \leq |\beta| \leq 5$ w.r.t. the local Maxwellian $\mathbf{M} = \mathbf{M}_{[\rho,u,\theta]}$. First, notice that from (4.3) and (4.4) we have

$$\begin{aligned}\sup_{\tau\in[0,t],x\in\mathbf{R}}\left\{\sum_{0\leq|\alpha|\leq 3}\left\|\frac{\sqrt{1+|\xi|}\partial^\alpha \mathbf{G}}{\sqrt{\mathbf{M}}}\right\|_{L^2_\xi}\right\} &\leq O(1)\sup_{\tau\in[0,t],x\in\mathbf{R}}\left\{\sum_{0\leq|\alpha|\leq 3}\left\|\frac{\partial^\alpha \mathbf{G}}{\sqrt{\mathbf{M}_-}}\right\|_{L^2_\xi}\right\} \\ &\leq O(1)(\varepsilon+\delta_0).\end{aligned} \tag{4.33}$$

For estimates on $\partial^\alpha \mathbf{M}$, applying $\mathbf{P}_0$ to (1.8) gives

$$\begin{aligned}&\mathbf{M}_t + \mathbf{P}_0(\xi_1\partial_x \mathbf{M}) + \mathbf{P}_0\left\{\xi_1\partial_x\left[L^{-1}_{\mathbf{M}}(\mathbf{P}_1(\xi_1\partial_x\mathbf{M}))\right]\right\} \\ &= -\mathbf{P}_0\left\{\xi_1\partial_x\left[L^{-1}_{\mathbf{M}}(\mathbf{G}_t+\mathbf{P}_1(\xi_1\partial_x\mathbf{G}) - Q(\mathbf{G},\mathbf{G}))\right]\right\}.\end{aligned} \tag{4.34}$$

Applying $\partial^\alpha(1\leq|\alpha|\leq 4)$ to (4.34) and integrating its product with $\frac{\partial^\alpha \mathbf{M}}{\mathbf{M}}$ over $[0,t]\times\mathbf{R}\times\mathbf{R}^3$ yield

$$\begin{aligned}&\frac{1}{2}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha\mathbf{M}|^2}{\mathbf{M}}d\xi dx\Big|_{\tau=0}^{\tau=t} = -\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{|\partial^\alpha\mathbf{M}|^2}{\mathbf{M}^2}\mathbf{M}_t + \frac{\partial^\alpha\mathbf{M}}{\mathbf{M}}\partial^\alpha[\mathbf{P}_0(\xi_1\partial_x\mathbf{M})]\right)d\xi dx d\tau \\ &-\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^\alpha\mathbf{M}\partial^\alpha\left\{\mathbf{P}_0\left\{\xi_1\partial_x\left[L^{-1}_{\mathbf{M}}(\mathbf{P}_1(\xi_1\partial_x\mathbf{M}))\right]\right\}\right\}}{\mathbf{M}}d\xi dx d\tau \\ &-\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^\alpha\mathbf{M}\partial^\alpha\left\{\mathbf{P}_0\left\{\xi_1\partial_x\left[L^{-1}_{\mathbf{M}}(\mathbf{G}_t+\mathbf{P}_1(\xi_1\partial_x\mathbf{G})-Q(\mathbf{G},\mathbf{G}))\right]\right\}\right\}}{\mathbf{M}}d\xi dx d\tau \\ &:= \sum_{j=5}^{7} I_j.\end{aligned} \tag{4.35}$$

In the following, we estimate $I_j (j=5,6,7)$ term by term. First, (4.3) and properties of the operator $\mathbf{P}_0$ and $\mathbf{P}_1$ give

$$|I_5| \leq O(1)\varepsilon^{\frac{1}{8}} + O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\left(\left|(\tilde{\rho}_x,\tilde{u}_x,\tilde{\theta}_x)\right|^2 + \sum_{1<|\beta|\leq|\alpha|+1}\left|\partial^\beta(\rho,u,\theta)\right|^2\right)dx d\tau. \tag{4.36}$$

Since

$$\begin{aligned}I_6 = &-\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial^\alpha\mathbf{M})\partial^\alpha\left\{\mathbf{P}_0\left\{\xi_1\partial_x\left[L^{-1}_{\mathbf{M}}(\mathbf{P}_1(\xi_1\partial_x\mathbf{M}))\right]\right\}\right\}}{\mathbf{M}}d\xi dx d\tau \\ &-\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\partial^\alpha\mathbf{M})\partial^\alpha\left\{\mathbf{P}_0\left\{\xi_1\partial_x\left[L^{-1}_{\mathbf{M}}(\mathbf{P}_1(\xi_1\partial_x\mathbf{M}))\right]\right\}\right\}}{\mathbf{M}}d\xi dx d\tau \\ := &\, I_6^1 + I_6^2,\end{aligned} \tag{4.37}$$

and

$$\partial^\alpha\left\{L^{-1}_{\mathbf{M}}h\right\} = L^{-1}_{\mathbf{M}}(\partial^\alpha h) - \sum_{j=0}^{|\alpha|-1}\sum_{|\alpha_j|=j}C_{\alpha_j}L^{-1}_{\mathbf{M}}\left(Q\left(\partial^{\alpha_j}\left(L^{-1}_{\mathbf{M}}h\right),\partial^{\alpha-\alpha_j}\mathbf{M}\right)\right)$$

where C_{α_j} are some positive constants. Thus,

$$
\begin{aligned}
I_6^1 &= -\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial^\alpha\mathbf{M})\partial^\alpha\left\{\xi_1\partial_x\left[L_{\mathbf{M}}^{-1}(\mathbf{P}_1(\xi_1\partial_x\mathbf{M}))\right]\right\}}{\mathbf{M}}d\xi dxd\tau\\
&= -\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1[\xi_1\partial_x(\mathbf{P}_0(\partial^\alpha\mathbf{M}))]\partial^\alpha\left\{L_{\mathbf{M}}^{-1}[\mathbf{P}_1(\xi_1\partial_x\mathbf{M})]\right\}}{\mathbf{M}}d\xi dxd\tau\\
&\quad+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1[\xi_1\mathbf{P}_0(\partial^\alpha\mathbf{M})]\partial^\alpha\left\{L_{\mathbf{M}}^{-1}[\mathbf{P}_1(\xi_1\partial_x\mathbf{M})]\right\}}{\mathbf{M}^2}\mathbf{M}_x d\xi dxd\tau\\
&\leq -d_4\int_0^t\int_{\mathbf{R}}\left|\partial^\alpha(u_x,\theta_x)\right|^2dxd\tau+O(1)\varepsilon^{\frac{1}{8}}\\
&\quad+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\left(\left|(\tilde{\rho}_x,\tilde{u}_x,\tilde{\theta}_x)\right|^2+\sum_{1<|\beta|\leq|\alpha|}\left|\partial^\beta(\rho,u,\theta)\right|^2\right)dxd\tau,
\end{aligned}
\tag{4.38}
$$

and

$$
\begin{aligned}
I_6^2 &= -\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^{\beta'}(\mathbf{P}_1(\partial^\alpha\mathbf{M}))\partial^{\alpha-\beta'}\left\{\mathbf{P}_0\left\{\xi_1\partial_x\left[L_{\mathbf{M}}^{-1}(\mathbf{P}_1(\xi_1\partial_x\mathbf{M}))\right]\right\}\right\}}{\mathbf{M}}d\xi dxd\tau\\
&\quad+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\mathbf{P}_1(\partial^\alpha\mathbf{M})\partial^{\alpha-\beta'}\left\{\mathbf{P}_0\left\{\xi_1\partial_x\left[L_{\mathbf{M}}^{-1}(\mathbf{P}_1(\xi_1\partial_x\mathbf{M}))\right]\right\}\right\}}{\mathbf{M}^2}\partial^{\beta'}\mathbf{M}d\xi dxd\tau\\
&\leq \frac{d_4}{3}\int_0^t\int_{\mathbf{R}}\left|\partial^\alpha(u_x,\theta_x)\right|^2dxd\tau+O(1)\varepsilon^{\frac{1}{8}}\\
&\quad+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\left(\left|(\tilde{\rho}_x,\tilde{u}_x,\tilde{\theta}_x)\right|^2+\sum_{1<|\beta|\leq|\alpha|}\left|\partial^\beta(\rho,u,\theta)\right|^2\right)dxd\tau.
\end{aligned}
\tag{4.39}
$$

Here d_4 is a positive constant, and $\beta'=(1,0)$ or $(0,1)$ which depends on the special form of α.

Hence

$$
\begin{aligned}
I_6 &\leq -\frac{2d_4}{3}\int_0^t\int_{\mathbf{R}}\left|\partial^\alpha(u_x,\theta_x)\right|^2dxd\tau+O(1)\varepsilon^{\frac{1}{8}}\\
&\quad+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\left(\left|(\tilde{\rho}_x,\tilde{u}_x,\tilde{\theta}_x)\right|^2+\sum_{1<|\beta|\leq|\alpha|}\left|\partial^\beta(\rho,u,\theta)\right|^2\right)dxd\tau.
\end{aligned}
\tag{4.40}
$$

Finally, by Lemma 2.1, (4.2), (4.3) and choosing a sufficiently small constant $\lambda>0$, we have

$$
\begin{aligned}
|I_7| &\leq \lambda\int_0^t\int_{\mathbf{R}}\left|\partial^\alpha(u_x,\theta_x)\right|^2dxd\tau+O(1)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\left(|\partial^\alpha\mathbf{G}_x|^2+|\partial^\alpha\mathbf{G}_t|^2\right)}{\mathbf{M}}d\xi dxd\tau\\
&\quad+O(1)\varepsilon^{\frac{1}{8}}+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\left(\left|(\tilde{\rho}_x,\tilde{u}_x,\tilde{\theta}_x)\right|^2+\sum_{1<|\beta|\leq|\alpha|}\left|\partial^\beta(\rho,u,\theta)\right|^2\right)dxd\tau\\
&\quad+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}}+\sum_{1\leq|\beta|\leq|\alpha|}\frac{(1+|\xi|)|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}\right)d\xi dxd\tau.
\end{aligned}
\tag{4.41}
$$

Combining (4.35), (4.36), (4.40), and (4.41) gives

$$\begin{aligned}
&\sum_{1\le|\alpha|\le4}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha\mathbf{M}|^2}{\mathbf{M}}d\xi dx\Big|_{\tau=0}^{\tau=t}+\sum_{1\le|\alpha|\le4}\int_0^t\int_{\mathbf{R}}\left|\partial^\alpha(u_x,\theta_x)\right|^2dxd\tau\\
&\le O(1)\varepsilon^{\frac18}+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\left(\left|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)\right|^2+\sum_{1<|\beta|\le5}\left|\partial^\beta\rho\right|^2\right)dxd\tau\\
&+O(1)\sum_{|\alpha|=4}\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\big(|\partial^\alpha\mathbf{G}_x|^2+|\partial^\alpha\mathbf{G}_t|^2\big)}{\mathbf{M}}d\xi dxd\tau\\
&+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}}+\sum_{1\le|\beta|\le4}\frac{(1+|\xi|)|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}\right)d\xi dxd\tau.
\end{aligned}\tag{4.42}$$

Similar to the procedure to obtain (4.42), we have the following estimates on $\partial^\alpha\mathbf{G}$ by using the equation (1.10)

$$\begin{aligned}
&\sum_{1\le|\alpha|\le4}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dx\Big|_{\tau=0}^{\tau=t}+\sum_{1\le|\alpha|\le4}\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\le O(1)\varepsilon^{\frac18}+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\left(\left|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)\right|^2+\sum_{1<|\beta|\le4}\left|\partial^\beta\rho\right|^2\right)dxd\tau\\
&+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}}+\sum_{1\le|\beta|\le4}\frac{(1+|\xi|)|\partial^\beta\mathbf{G}|^2}{\mathbf{M}_-}\right)d\xi dxd\tau\\
&+O(1)\sum_{|\alpha|=4}\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\big(|\partial^\alpha\mathbf{G}_x|^2+|\partial^\alpha\mathbf{G}_t|^2\big)}{\mathbf{M}}d\xi dxd\tau\\
&+O(1)\sum_{1\le|\alpha|\le4}\int_0^t\int_{\mathbf{R}}\left|\partial^\alpha(u_x,\theta_x)\right|^2dxd\tau.
\end{aligned}\tag{4.43}$$

For the estimate on $\partial^\alpha f$, applying $\partial^\alpha(2\le|\alpha|\le5)$ to (1.1) and integrating its product with $\frac{\partial^\alpha f}{\mathbf{M}}$ over $[0,t]\times\mathbf{R}\times\mathbf{R}^3$ to obtain

$$\begin{aligned}
&\frac12\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha f|^2}{\mathbf{M}}d\xi dx\Big|_{\tau=0}^{\tau=t}=-\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha f|^2}{\mathbf{M}^2}\left(\mathbf{M}_t+\xi_1\mathbf{M}_x\right)d\xi dxd\tau\\
&\quad+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^\alpha f\partial^\alpha(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi dxd\tau+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^\alpha f\partial^\alpha(Q(\mathbf{G},\mathbf{G}))}{\mathbf{M}}d\xi dxd\tau\\
&:=\sum_{j=8}^{10}I_j.
\end{aligned}\tag{4.44}$$

It is easy to see that

$$|I_8|\le O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha f|^2}{\mathbf{M}_-}d\xi dxd\tau.\tag{4.45}$$

For I_9, since

$$\begin{aligned}
I_9&=\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^\alpha\mathbf{G}\partial^\alpha(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi dxd\tau+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\partial^\alpha\mathbf{M}\partial^\alpha(L_{\mathbf{M}}\mathbf{G})}{\mathbf{M}}d\xi dxd\tau\\
&:=I_9^1+I_9^2,
\end{aligned}\tag{4.46}$$

we have from (1.14), Lemma 2.1, (4.3), (4.4) that

$$
\begin{aligned}
I_9^1 &= \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\partial^\alpha \mathbf{G} L_{\mathbf{M}}(\partial^\alpha \mathbf{G})}{\mathbf{M}} d\xi dx d\tau \\
&\quad + 2 \sum_{j=0}^{|\alpha|-1} \sum_{|\alpha_j|=j} C_{\alpha_j} \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\partial^\alpha \mathbf{G} Q\big(\partial^{\alpha_j}\mathbf{G}, \partial^{\alpha-\alpha_j}\mathbf{M}\big)}{\mathbf{M}} d\xi dx d\tau \\
&\le -\frac{\sigma}{2} \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{(1+|\xi|)|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\
&\quad + O(1) \sum_{j=0}^{|\alpha|-1} \sum_{|\alpha_j|=j} \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{(1+|\xi|)^{-1} Q\big(\partial^{\alpha_j}\mathbf{G}, \partial^{\alpha-\alpha_j}\mathbf{M}\big)^2}{\mathbf{M}} d\xi dx d\tau \\
&\le -\frac{\sigma}{2} \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{(1+|\xi|)|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\
&\quad + O(1) \sum_{j=0}^{|\alpha|-1} \sum_{|\alpha_j|=j} \int_0^t \int_{\mathbf{R}} \left(\int_{\mathbf{R}^3} \frac{(1+|\xi|)\big|\partial^{\alpha_j}\mathbf{G}\big|^2}{\mathbf{M}} d\xi \right) \cdot \left(\int_{\mathbf{R}^3} \frac{(1+|\xi|)\big|\partial^{\alpha-\alpha_j}\mathbf{M}\big|^2}{\mathbf{M}} d\xi \right) d\xi dx d\tau \\
&\le -\frac{\sigma}{2} \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{(1+|\xi|)|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\
&\quad + O(1)\varepsilon^{\frac{1}{8}} + O(1)(\varepsilon+\delta_0) \int_0^t \int_{\mathbf{R}} \left(\Big|(\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x)\Big|^2 + \sum_{1<|\beta|\le|\alpha|} \Big|\partial^\beta(\rho, u, \theta)\Big|^2 \right) dx d\tau \\
&\quad + O(1)(\varepsilon+\delta_0) \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}} + \sum_{2\le|\beta|\le|\alpha|} \frac{(1+|\xi|)|\partial^\beta \mathbf{G}|^2}{\mathbf{M}} \right) d\xi dx d\tau,
\end{aligned}
\tag{4.47}
$$

and

$$
\begin{aligned}
I_9^2 &= \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\mathbf{P}_1(\partial^\alpha \mathbf{M}) \partial^\alpha (L_{\mathbf{M}} G)}{\mathbf{M}} d\xi dx d\tau \\
&= 2 \sum_{j=0}^{|\alpha|-1} \sum_{|\alpha_j|=j} C_{\alpha_j} \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{\mathbf{P}_1(\partial^\alpha \mathbf{M}) Q\big(\partial^{\alpha_j}\mathbf{G}, \partial^{\alpha-\alpha_j}\mathbf{M}\big)}{\mathbf{M}} d\xi dx d\tau \\
&\le O(1)\varepsilon^{\frac{1}{8}} + O(1)(\varepsilon+\delta_0) \int_0^t \int_{\mathbf{R}} \left(\Big|(\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x)\Big|^2 + \sum_{1<|\beta|\le|\alpha|} \Big|\partial^\beta(\rho, u, \theta)\Big|^2 \right) dx d\tau \\
&\quad + O(1)(\varepsilon+\delta_0) \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}} + \sum_{2\le|\beta|\le|\alpha|} \frac{(1+|\xi|)|\partial^\beta \mathbf{G}|^2}{\mathbf{M}} \right) d\xi dx d\tau.
\end{aligned}
\tag{4.48}
$$

Hence

$$
\begin{aligned}
I_9 &\le -\frac{\sigma}{3} \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{(1+|\xi|)|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\
&\quad + O(1)\varepsilon^{\frac{1}{8}} + O(1)(\varepsilon+\delta_0) \int_0^t \int_{\mathbf{R}} \left(\Big|(\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x)\Big|^2 + \sum_{1<|\beta|\le|\alpha|} \Big|\partial^\beta(\rho, u, \theta)\Big|^2 \right) dx d\tau \\
&\quad + O(1)(\varepsilon+\delta_0) \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}} + \sum_{2\le|\beta|\le|\alpha|} \frac{(1+|\xi|)|\partial^\beta \mathbf{G}|^2}{\mathbf{M}} \right) d\xi dx d\tau.
\end{aligned}
\tag{4.49}
$$

Similarly, we have for I_{10}

$$
\begin{aligned}
|I_{10}| &\le O(1)\varepsilon^{\frac{1}{8}} + O(1)(\varepsilon+\delta_0) \int_0^t \int_{\mathbf{R}} \left(\Big|(\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x)\Big|^2 + \sum_{1<|\beta|\le|\alpha|} \Big|\partial^\beta(\rho, u, \theta)\Big|^2 \right) dx d\tau \\
&\quad + O(1)(\varepsilon+\delta_0) \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}} + \sum_{2\le|\beta|\le|\alpha|} \frac{(1+|\xi|)|\partial^\beta \mathbf{G}|^2}{\mathbf{M}} \right) d\xi dx d\tau \\
&\quad + \lambda \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{(1+|\xi|)|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau,
\end{aligned}
\tag{4.50}
$$

where $\lambda > 0$ is a sufficiently small positive constant.

Hence, (4.44)-(4.50) give

$$\begin{aligned}
&\sum_{2\leq|\alpha|\leq 5}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha f|^2}{\mathbf{M}}d\xi dx\Big|_{\tau=0}^{\tau=t}+\sum_{2\leq|\alpha|\leq 5}\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau\\
&\leq O(1)\varepsilon^{\frac18}+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\left(\left|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)\right|^2+\sum_{1<|\beta|\leq 5}\left|\partial^\beta(\rho,u,\theta)\right|^2\right)dxd\tau\\
&\quad+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}}+\sum_{2\leq|\beta|\leq 5}\frac{(1+|\xi|)|\partial^\beta f|^2}{\mathbf{M}_-}\right)d\xi dxd\tau.
\end{aligned}\tag{4.51}$$

By suitably linearly combining (4.42), (4.43) and (4.51), we have by choosing ε and δ_0 sufficiently small that

$$\begin{aligned}
&\sum_{1\leq|\alpha|\leq 4}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha \mathbf{G}|^2+|\partial^\alpha \mathbf{M}|^2}{\mathbf{M}}d\xi dx\Big|_{\tau=0}^{\tau=t}+\sum_{2\leq|\alpha|\leq 5}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha f|^2}{\mathbf{M}}d\xi dx\Big|_{\tau=0}^{\tau=t}\\
&+\sum_{1\leq|\alpha|\leq 5}\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)|\partial^\alpha \mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau+\sum_{1\leq|\alpha|\leq 4}\int_0^t\int_{\mathbf{R}}|\partial^\alpha(u_x,\theta_x)|^2dxd\tau\\
&\leq O(1)\varepsilon^{\frac18}+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\left(\left|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)\right|^2+\sum_{1<|\beta|\leq 5}\left|\partial^\beta\rho\right|^2\right)dxd\tau\\
&\quad+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}_-}+\sum_{1\leq|\beta|\leq 5}\frac{(1+|\xi|)|\partial^\beta \mathbf{G}|^2}{\mathbf{M}_-}\right)d\xi dxd\tau.
\end{aligned}\tag{4.52}$$

To recover the estimate on $|\partial^\alpha\rho|^2$ with $|\alpha|=5$ in (4.52), we need to use the conservation law (1.9), cf. [15] where the same technique was used. Since

$$\begin{cases}\rho_t+(\rho u_1)_x=0,\\ u_{1t}+u_1u_{1x}+\frac23\theta_x+\frac{2\theta}{3\rho}\rho_x+\int_{\mathbf{R}^3}\frac{\xi_1^2\mathbf{G}_x}{\rho}d\xi=0.\end{cases}\tag{4.53}$$

Now, for $1\leq|\alpha|\leq 4$, we consider

$$\int_0^t\int_{\mathbf{R}}(\partial^\alpha\rho_x)\,\partial^\alpha\left(u_{1t}+u_1u_{1x}+\frac23\theta_x+\frac{2\theta}{3\rho}\rho_x+\int_{\mathbf{R}^3}\frac{\xi_1^2\mathbf{G}_x}{\rho}d\xi\right)dxd\tau=0.\tag{4.54}$$

By (4.53), we have

$$\begin{aligned}
\int_0^t\int_{\mathbf{R}}|\partial^\alpha\rho_x|^2dxd\tau\leq\ & O(1)\varepsilon^{\frac18}+O(1)\int_{\mathbf{R}}\left(\sum_{|\beta|=|\alpha|+1}\left|\partial^\beta\rho\right|^2+\sum_{|\beta|=|\alpha|}\left|\partial^\beta u\right|^2\right)(t)dx\\
&+O(1)\int_{\mathbf{R}}\left(\sum_{|\beta|=|\alpha|+1}\left|\partial^\beta\rho\right|^2+\sum_{|\beta|=|\alpha|}\left|\partial^\beta u\right|^2\right)(0)dx\\
&+O(1)\sum_{|\beta|=|\alpha|+1}\int_{\mathbf{R}}\left(\left|\partial^\beta(u,\theta)\right|^2+\int_{\mathbf{R}^3}\frac{|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi\right)(t)dxd\tau\\
&+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\left(\left|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)\right|^2+\sum_{1<|\beta|\leq|\alpha|}\left|\partial^\beta(\rho,u,\theta)\right|^2\right)dxd\tau\\
&+O(1)(\varepsilon+\delta_0)\sum_{2\leq|\beta|\leq|\alpha|}\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi dxd\tau.
\end{aligned}\tag{4.55}$$

Thus

$$\begin{aligned}\sum_{1\le|\alpha|\le4}\int_0^t\int_{\mathbf{R}}|\partial^\alpha\rho_x|^2\,dxd\tau\le{}&O(1)\int_{\mathbf{R}}\Big(\sum_{1\le|\beta|\le5}\left|\partial^\beta\rho\right|^2+\sum_{1\le|\beta|\le4}\left|\partial^\beta u\right|^2\Big)(t)dx\\&+O(1)\int_{\mathbf{R}}\Big(\sum_{1\le|\beta|\le5}\left|\partial^\beta\rho\right|^2+\sum_{1\le|\beta|\le4}\left|\partial^\beta u\right|^2\Big)(0)dx\\&+O(1)\varepsilon^{\frac18}+O(1)\int_0^t\int_{\mathbf{R}}\Big((\varepsilon+\delta_0)\left|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)\right|^2\\&+\sum_{1<|\beta|\le5}\Big(\left|\partial^\beta(u,\theta)\right|^2+\int_{\mathbf{R}^3}\tfrac{|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi\Big)\Big)dxd\tau.\end{aligned}\tag{4.56}$$

Moreover, since

$$\begin{aligned}\int_{\mathbf{R}^3}\tfrac{|\partial^\beta f|^2}{\mathbf{M}}d\xi&=\int_{\mathbf{R}^3}\tfrac{|\partial^\beta\mathbf{M}+\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi=\int_{\mathbf{R}^3}\tfrac{|\partial^\beta\mathbf{M}|^2+2\partial^\beta\mathbf{M}\partial^\beta\mathbf{G}+|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi\\&=\int_{\mathbf{R}^3}\tfrac{\left|\mathbf{P}_0(\partial^\beta\mathbf{M})\right|^2+\left|\mathbf{P}_1(\partial^\beta\mathbf{M})\right|^2+2\mathbf{P}_1(\partial^\beta\mathbf{M})\partial^\beta\mathbf{G}+\left|\partial^\beta\mathbf{G}\right|^2}{\mathbf{M}}d\xi\\&\ge\int_{\mathbf{R}^3}\tfrac{\left|\mathbf{P}_0(\partial^\beta\mathbf{M})\right|^2}{\mathbf{M}}d\xi,\end{aligned}\tag{4.57}$$

and

$$\begin{aligned}\mathbf{P}_0(\partial^\alpha\mathbf{M})={}&\tfrac{\mathbf{M}}{\rho}\partial^\alpha\rho+\tfrac{\mathbf{M}}{2\theta}\left(\tfrac{|\xi-u|^2}{R\theta}-3\right)\partial^\alpha\theta+\tfrac{(\xi-u)\cdot\partial^\alpha u}{R\theta}\mathbf{M}\\&+\sum_{j=1}^{|\alpha|-1}\sum_{|\alpha_j|=j}C_{\alpha_j}\rho^{-1}\theta^{\frac32}\partial^{\alpha-\alpha_j}\left(\rho\theta^{-\frac32}\right)\partial^{\alpha_j}\left(-\tfrac{|\xi-u|^2}{2R\theta}\right)\mathbf{M}\\&+\sum_{j=1}^{|\alpha|-1}\sum_{|\alpha_j|=j}C_{\alpha_j}\left\{\rho^{-1}\theta^{\frac32}\partial^{\alpha-\alpha_j}\rho\partial^{\alpha_j}\left(\theta^{-\frac32}\right)-\partial^{\alpha-\alpha_j}(|\xi-u|^2)\partial^{\alpha_j}\left(\tfrac{1}{2R\theta}\right)\right\}\mathbf{M},\end{aligned}\tag{4.58}$$

By induction, from (1.5) we have

$$\sum_{1\le|\alpha|\le5}\int_{\mathbf{R}}|\partial^\alpha(\rho,u)|^2\,dx\le O(1)\sum_{1\le|\alpha|\le5}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha f|^2}{\mathbf{M}}d\xi dx.\tag{4.59}$$

(4.59) and (4.56) give

$$\begin{aligned}\sum_{1\le|\alpha|\le4}\int_0^t\int_{\mathbf{R}}|\partial^\alpha\rho_x|^2\,dxd\tau\le{}&O(1)\left(\varepsilon^{\frac18}+N(0)^2\right)+O(1)\int_0^t\int_{\mathbf{R}}\Big((\varepsilon+\delta_0)\left|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)\right|^2\\&+\sum_{1<|\beta|\le5}\Big(\left|\partial^\beta(u,\theta)\right|^2+\int_{\mathbf{R}^3}\tfrac{|\partial^\beta\mathbf{G}|^2}{\mathbf{M}}d\xi\Big)\Big)dxd\tau\\&+O(1)\sum_{1\le|\alpha|\le5}\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\tfrac{|\partial^\alpha f|^2}{\mathbf{M}}d\xi dxd\tau.\end{aligned}\tag{4.60}$$

Finally, from (4.27), (4.31), (4.32), (4.52), and (4.60), we have

$$\begin{aligned}
&\int_{\mathbf{R}}\eta(t)d\xi+\int_0^t\int_{\mathbf{R}}\left(|(\tilde{\rho}_x,\tilde{u}_x,\tilde{\theta}_x)|^2+\bar{u}_{1x}|(\tilde{\rho},\tilde{u},\tilde{\theta})|^2+\sum_{1<|\beta|\le 5}\left|\partial^\beta(\rho,u,\theta)\right|^2\right)dxd\tau\\
&\quad+\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}}d\xi d\tau+\sum_{1\le|\alpha|\le 4}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha\mathbf{M}|^2+|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}d\xi dx+\sum_{|\alpha|=5}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha f|^2}{\mathbf{M}}d\xi dx\\
&\quad+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}}+\sum_{1\le|\alpha|\le 5}\frac{(1+|\xi|)|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}}\right)d\xi dxd\tau\\
&\le O(1)\left(\varepsilon^{\frac{1}{8}}+N(0)^2\right)+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-}+\sum_{1\le|\alpha|\le 5}\frac{|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}_-}\right)d\xi dxd\tau.
\end{aligned}\tag{4.61}$$

Here we have used the fact that

$$\int_{\mathbf{R}^3}\frac{f^2}{\mathbf{M}}d\xi\le O(1)\int_{\mathbf{R}^3}\frac{f^2}{\mathbf{M}_-}d\xi,$$

for $\theta_-<\theta$. A direct consequence of (4.61) gives

$$\begin{aligned}
&\int_0^t\int_{\mathbf{R}}\left(|(\tilde{\rho}_x,\tilde{u}_x,\tilde{\theta}_x)|^2+\sum_{1<|\beta|\le 5}\left|\partial^\beta(\rho,u,\theta)\right|^2\right)dxd\tau\\
&\le O(1)\left(\varepsilon^{\frac{1}{8}}+N(0)^2\right)+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-}+\sum_{1\le|\alpha|\le 5}\frac{|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}_-}\right)d\xi dxd\tau.
\end{aligned}\tag{4.62}$$

4.3 Higher order estimates w.r.t. $\mathbf{M}_-$

In this subsection, we will consider certain higher order energy estimates w.r.t. the global Maxwellian $\mathbf{M}_-=\mathbf{M}_{[\rho_-,u_-,\theta_-]}$. For brevity of presentation, we will only point out the main differences and give the corresponding estimates without proving them in details.

The first main difference is that the fluid part $\mathbf{P}_0(\partial^\alpha\mathbf{M})$ and the non-fluid part $\mathbf{G}$ are no longer orthogonal w.r.t. the global Maxwellian $\mathbf{M}_-$, i.e.

$$\int_{\mathbf{R}^3}\frac{\mathbf{P}_0(\partial^\alpha\mathbf{M})\cdot\mathbf{G}}{\mathbf{M}_-}d\xi\neq 0.$$

As a result, there is an extra error term in the form of

$$O(1)\int_0^t\int_{\mathbf{R}}\left(\left|(\tilde{\rho}_x,\tilde{u}_x,\tilde{\theta}_x)\right|^2+\sum_{2\le|\alpha|\le 5}|\partial^\alpha(\rho,u,\theta)|^2\right)dxd\tau.$$

The second difference is that the corresponding *a priori* estimate similar to (4.33), i.e.

$$\sup_{\tau\in[0,t],x\in\mathbf{R}}\left\{\sum_{0\le|\alpha|\le 3}\left\|\frac{\sqrt{1+|\xi|}\partial^\alpha\mathbf{G}}{\sqrt{\mathbf{M}_-}}\right\|_{L^2_\xi}\right\}\le O(1)(\varepsilon+\delta_0),$$

does not hold. Then the estimate on the derivatives on $Q(\mathbf{G},\mathbf{G})$ is different without the weight $\sqrt{1+|\xi|}$ in the *a priori* estimate. For illustration, we give the estimate on the following typical term when we differentiate $Q(\mathbf{G},\mathbf{G})$ after applying Lemma 2.1.

$$\begin{aligned}
I_{11}&=\sum_{j=0}^{|\alpha|}\sum_{|\alpha_j|=j}\int_0^t\int_{\mathbf{R}}\left(\int_{\mathbf{R}^3}\frac{(1+|\xi|)\left|\partial^{\alpha_j}\mathbf{G}\right|^2}{\mathbf{M}_-}d\xi\right)\cdot\left(\int_{\mathbf{R}^3}\frac{\left|\partial^{\alpha-\alpha_j}\mathbf{G}\right|^2}{\mathbf{M}_-}d\xi\right)d\xi dxd\tau\\
&:=\sum_{j=0}^{|\alpha|}\sum_{|\alpha_j|=j}I_{11}^{\alpha_j}.
\end{aligned}\tag{4.63}$$

Here $1 \leq |\alpha| \leq 5$.

If $|\alpha - \alpha_j| \leq 3$, then we have from (4.3) that

$$
\begin{aligned}
I_{11}^{\alpha_j} \leq & O(1)\varepsilon^{\frac{1}{8}} + O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}} |(\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x)|^2 dx d\tau \\
& + O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}_-} + \sum_{1\leq|\beta|\leq|\alpha_j|}\frac{(1+|\xi|)|\partial^\beta \mathbf{G}|^2}{\mathbf{M}_-}\right) d\xi dx d\tau.
\end{aligned}
$$

If $|\alpha - \alpha_j| \geq 4$, the above analysis does not apply because (4.3) holds only for $|\alpha| \leq 3$. However, from (4.57), (4.58), and (4.59), we have

$$
\begin{aligned}
\sum_{|\alpha|=5}\left\|\frac{\partial^\alpha \mathbf{G}}{\sqrt{\mathbf{M}_-}}\right\|_{L_x^2(L_\xi^2)} & \leq O(1)\sum_{|\alpha|=5}\left(\left\|\frac{\partial^\alpha f}{\sqrt{\mathbf{M}_-}}\right\|_{L_x^2(L_\xi^2)} + \left\|\frac{\partial^\alpha \mathbf{M}}{\sqrt{\mathbf{M}_-}}\right\|_{L_x^2(L_\xi^2)}\right) \\
& \leq O(1)\sum_{1\leq|\alpha|\leq 5}\left\|\frac{\partial^\alpha f}{\sqrt{\mathbf{M}_-}}\right\|_{L_x^2(L_\xi^2)} \\
& \leq O(1)(\varepsilon+\delta_0).
\end{aligned} \tag{4.64}
$$

With (4.64), we now estimate $I_{11}^{\alpha_j}$ for the case $|\alpha - \alpha_j| \geq 4$ as follows. Since

$$
\int_{\mathbf{R}^3}\frac{(1+|\xi|)\left|\partial^{\alpha_j}\mathbf{G}\right|^2}{\mathbf{M}_-}d\xi \leq O(1)\left(\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\left|\partial^{\alpha_j}\mathbf{G}\right|^2}{\mathbf{M}_-}d\xi dx\right)^{\frac{1}{2}}\left(\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\left|\partial^{\alpha_j}\mathbf{G}_x\right|^2}{\mathbf{M}_-}d\xi dx\right)^{\frac{1}{2}},
$$

we have for $|\alpha_j| = 1$

$$
\begin{aligned}
I_{11}^{\alpha_j} & \leq O(1)\sup_{\tau\in[0,t]}\left\{\left\|\frac{\partial^{\alpha-\alpha_j}\mathbf{G}}{\sqrt{\mathbf{M}_-}}\right\|^2_{L_x^2(L_\xi^2)}\right\}\int_0^t\prod_{k=0}^{1}\left\|\frac{\sqrt{1+|\xi|}\partial^{\alpha_j}\partial_x^k\mathbf{G}}{\sqrt{\mathbf{M}_-}}\right\|_{L_x^2(L_\xi^2)} d\tau \\
& \leq O(1)(\varepsilon+\delta_0)^2\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\left(|\partial^{\alpha_j}\mathbf{G}|^2 + |\partial^{\alpha_j}\mathbf{G}_x|^2\right)}{\mathbf{M}_-}d\xi dx d\tau.
\end{aligned}
$$

Notice that $\overline{\mathbf{G}}$ is not in $L_x^2(L_\xi^2)$, when $\alpha_j = 0$, we need to factor out the supremum of L_ξ^2 norm of $\overline{\mathbf{G}}$ when it appears. Besides this, similar to the case when $\alpha_j = 1$, we have

$$
\begin{aligned}
I_{11}^{\alpha_j} & \leq O(1)\int_0^t\left\|\frac{\partial^\alpha \mathbf{G}}{\sqrt{\mathbf{M}_-}}\right\|^2_{L_x^2(L_\xi^2)}\prod_{k=0}^{1}\left\|\frac{\sqrt{1+|\xi|}\partial_x^k\mathbf{G}}{\sqrt{\mathbf{M}_-}}\right\|_{L_x^2(L_\xi^2)} d\tau \\
& \leq O(1)\int_0^t\left\|\frac{\partial^\alpha \mathbf{G}}{\sqrt{\mathbf{M}_-}}\right\|^2_{L_x^2(L_\xi^2)}\left\|\frac{\sqrt{1+|\xi|}\partial_x\mathbf{G}}{\sqrt{\mathbf{M}_-}}\right\|_{L_x^2(L_\xi^2)} \\
& \quad \times\left(\left\|\frac{\sqrt{1+|\xi|}\tilde{\mathbf{G}}}{\sqrt{\mathbf{M}_-}}\right\|_{L_x^2(L_\xi^2)} + \left\|\frac{\sqrt{1+|\xi|}\overline{\mathbf{G}}}{\sqrt{\mathbf{M}_-}}\right\|_{L_x^2(L_\xi^2)}\right) d\tau \\
& \leq O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\left(\tilde{\mathbf{G}}^2 + \sum_{1\leq|\beta|\leq|\alpha|}\left|\partial^\beta\mathbf{G}\right|^2\right)}{\mathbf{M}_-}d\xi dx d\tau.
\end{aligned}
$$

In summary, we have the estimate on (4.63) which is similar to (4.51).

Noticing the above two differences, similar to the estimate with the weight function

$\mathbf{M}$, we have the following estimate on derivatives with the weight function $\mathbf{M}_-$.

$$\begin{aligned}
&\sum_{1\leq|\alpha|\leq 4}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha\mathbf{G}|^2+|\partial^\alpha\mathbf{M}|^2}{\mathbf{M}_-}d\xi dx\Big|_{\tau=0}^{\tau=t}+\sum_{2\leq|\alpha|\leq 5}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha f|^2}{\mathbf{M}_-}d\xi dx\Big|_{\tau=0}^{\tau=t}\\
&+\sum_{1\leq|\alpha|\leq 5}\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx d\tau\\
&\leq O(1)\varepsilon^{\frac18}+O(1)\int_0^t\int_{\mathbf{R}}\left(\left|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)\right|^2+\sum_{1\leq|\alpha|\leq 4}|\partial^\alpha(\rho_x,u_x,\theta_x)|^2\right)dxd\tau\\
&+O(1)(\varepsilon+\delta_0)\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}_-}d\xi dx d\tau.
\end{aligned}\tag{4.65}$$

Finally, from (4.28), (4.62) and (4.65), we have the energy estimate

$$\begin{aligned}
&\int_{\mathbf{R}}\eta(t)d\xi+\int_0^t\int_{\mathbf{R}}\left(|(\tilde\rho_x,\tilde u_x,\tilde\theta_x)|^2+\overline{u}_{1x}|(\tilde\rho,\tilde u,\tilde\theta)|^2+\sum_{1<|\beta|\leq 5}\left|\partial^\beta(\rho,u,\theta)\right|^2\right)dxd\tau\\
&+\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_-}d\xi d\tau+\sum_{1\leq|\alpha|\leq 4}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha\mathbf{M}|^2+|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}_-}d\xi dx+\sum_{|\alpha|=5}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{|\partial^\alpha f|^2}{\mathbf{M}_-}d\xi dx\\
&+\int_0^t\int_{\mathbf{R}}\int_{\mathbf{R}^3}\left(\frac{(1+|\xi|)\tilde{\mathbf{G}}^2}{\mathbf{M}_-}+\sum_{1\leq|\alpha|\leq 5}\frac{(1+|\xi|)|\partial^\alpha\mathbf{G}|^2}{\mathbf{M}_-}\right)d\xi dx d\tau\\
&\leq O(1)\left(\varepsilon^{\frac18}+N(0)^2\right),
\end{aligned}\tag{4.66}$$

which close the *a priori* estimate (4.2) provided that we choose $\varepsilon_0>0$ and $\varepsilon>0$ sufficiently small such that

$$\begin{cases} N(0)<\varepsilon_0,\\ O(1)\left(\varepsilon^{\frac18}+\varepsilon_0^2\right)<\delta_0^2.\end{cases}$$

(4.66) and the convergence property on $(\overline{\rho},\overline{u},\overline{\theta})$ to (ρ^R,u^R,θ^R) in Section 3 give the global existence and the time asymptotic behavior of the solution stated in Theorem 1.1.

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