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Boltzmann Equation: Micro-Macro Decompositions and Positivity of Shock Profiles*

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Abstract: We introduce an elementary energy method for the Boltzmann equation based on a decomposition of the equation into macroscopic and microscopic components. The decomposition is useful for the study of time-asymptotic stability of nonlinear waves. The wave location is determined by the macroscopic equation. The microscopic component has an equilibrating property. The coupling of macroscopic and microscopic components gives rise naturally to the dissipations similar to those obtained by the Chapman-Enskog expansion. Our main result is the establishment of the positivity of shock profiles for the Boltzmann equation. This is shown by the time-asymptotic approach and the maximal principle for the collision operator.

Contents

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1. Introduction

Consider the Boltzmann equation, [2],

$$F_t + \xi \cdot \nabla_x F = Q(F, F). \tag{1.1}$$

The goal of the present paper is, first, to introduce decompositions of the Boltzmann equation into microscopic and macroscopic parts. We then devise an energy method for the study of time-asymptotic stability of nonlinear waves. Our main result is the establishment of the positivity of shock profiles for the Boltzmann equation using the time-asymptotic approach.

As usual, the macro-micro decomposition of a solution to the Boltzmann equation is made with respect to local Maxwellian states, [14]:

$$\omega(\xi; u_0, T_0) \equiv \frac{e^{-\frac{|\xi - u_0|^2}{2T_0}}}{\sqrt{(2\pi T_0)^3}} \text{ for } u_0, \xi \in \boldsymbol{R}^3; \text{ and } T_0 \in \boldsymbol{R}^+,$$

where the macroscopic velocity u_0, and temperature T_0 of the local thermal equilibrium state ω may be varying. Consider the perturbation, as customarily expressed, [8, 4, 7]:

$$F(x, t, \xi) \equiv \omega(\xi; u_0, T_0) + \omega^{\frac{1}{2}}(\xi; u_0, T_0)\, f(x, t, \xi). \tag{1.2}$$

Here ρ_0 is the macroscopic density. The collision invariants:

$$\begin{cases} \chi_0(\xi) \equiv 1, \\ \chi_i(\xi) \equiv \dfrac{\xi^i - u_0^i}{\sqrt{T_0}} \text{ for } i = 1, 2, 3, \\ \chi_4(\xi) \equiv \dfrac{1}{\sqrt{6}}\left(\dfrac{|\xi - u_0|^2}{T_0} - 3\right), \\ \displaystyle\int_{\boldsymbol{R}^3} \chi_i(\xi) Q(F, F)(\xi) d\xi = 0,\ i = 0, \cdots, 4, \end{cases}$$

are normalized with respect to the Maxwellian $\omega(\xi; u_0, T_0)$;

$$\int_{\boldsymbol{R}^3} \omega(\xi; u_0, T_0) \chi_i(\xi) \chi_j(\xi) d\xi = \delta_{ij},\ i, j = 0, \cdots, 4.$$

The perturbation is decomposed into the macroscopic and microscopic parts. The macroscopic part f_0 is in the range of the projection operator $\boldsymbol{P}_0$ on the space spanned by $\chi_i \omega^{1/2}$, $i = 0, \cdots, 4$,; and the microscopic part is in the range of the orthogonal projection $\boldsymbol{P}_1$:

$$\begin{cases} f(x, t, \xi) = f_0(x, t, \xi) + f_1(x, t, \xi), \\ f_0 = \boldsymbol{P}_0 f,\ f_1 = \boldsymbol{P}_1 f, \\ f_0(x, t, \xi) \equiv \left(\rho(x, t) + \displaystyle\sum_{i=1}^{3} m^i(x, t) \chi_i(\xi) + e(x, t) \chi_4(\xi)\right) \omega^{\frac{1}{2}}(\xi; u_0, T_0). \end{cases}$$

The linearized collision operator

$$L(h) \equiv \omega^{-\frac{1}{2}} \, [Q(\omega, \omega^{\frac{1}{2}} h) + Q(\omega^{\frac{1}{2}} h, \omega)],$$

has two basic properties: The macroscopic part is in the null space:

$$L f_0 = 0.$$

Another basic property, studied by [4] for the hard sphere and the refinement of cut-off potentials by [7], is that the linearized collision operator is negative definite on the microscopic part:

$$\int_{\boldsymbol{R}^3} f_1 L f_1 d\xi < -\nu_0 \int_{\boldsymbol{R}^3} (f_1)^2 d\xi \tag{1.3}$$

for some positive constant ν_0. The Boltzmann equation is decomposed into macroscopic and microscopic equations, cf. (2.10):

$$\begin{cases} f_{0t} + \boldsymbol{P}_0 \xi \cdot \nabla_x (f_0 + f_1) = 0, \\ f_{1t} + \boldsymbol{P}_1 \xi \cdot \nabla_x (f_0 + f_1) = (1+\rho) L f_1 + \boldsymbol{N}(f). \end{cases} \tag{1.4}$$

Here, (1.3) is a dissipative mechanism driving the non-equilibrium state towards the equilibrium state. It resembles the Boltzmann H-theorem.

We will apply the basic energy method to both equations. Our energy method will yield naturally the following expression of the dissipation for the macroscopic part:

$$\begin{aligned} &-\boldsymbol{P}_0 \xi \cdot \nabla_x L^{-1} \boldsymbol{P}_1 (\xi \cdot \nabla_x f_0) \\ &\quad = -\boldsymbol{P}_0 \xi \cdot \nabla_x L^{-1} \left(\boldsymbol{P}_1 \sum_{i=1}^{3} (\xi \cdot \nabla_x m^i) \, \chi_i + \boldsymbol{P}_1 (\xi \cdot \nabla_x e) \, \chi_4 \right). \end{aligned} \tag{1.5}$$

The right-hand side of this identity represents the dissipation of the momentum and the energy, but not the mass. This is consistent with the compressible Navier-Stokes equations. The Navier-Stokes equations are an approximation to the Boltzmann equation through the Chapman-Enskog expansion, [5]. The above is an exact dissipation expression.

The Equilibrating Property (1.3) is used for the microscopic equation, while the dissipation expression (1.5) arises when we apply the energy method to the macroscopic equation. In the study of nonlinear waves, we make the decomposition around the local Maxwellian states corresponding to the waves. The macroscopic component determines the wave propagation. For instance, it dictates the phase location of the traveling waves. The microscopic component is the faster decaying part. The geometry of the wave dictates its stability and comes up naturally in the convection term of the macroscopic equation. These are the basic understandings when carrying out the energy estimates. In Sect. 2, we first briefly recall the basics of the Boltzmann equation and introduce the macro-micro decomposition. The elementary relationship between the macroscopic and microscopic components, particularly the dissipation expression (1.5) are presented in Sect. 3. Basic energy estimates are carried out in Sect. 4 for the linear equation. The nonlinear stability of the Maxwellian states is shown in Sect. 5.

The time-asymptotic stability of the Maxwellian states has been shown previously using a different energy method based on the Fourier transform and spectral analysis, [17, 18, 15, 10]. These studies yield stronger results with rates of convergence. Our thinking is that the energy method and the more quantitative method based on the spectral analysis, or equivalently, the Green functions, should be viewed as separate approaches. The present energy method is an elementary and basic approach. Our method is also being applied to the study of supersonic boundary layers, [19].

We then study the positivity of shock profiles. In [3] the profiles are constructed by solving the equation

$$-sF' + \xi_1 F' = Q(F, F).$$

Here s is the speed of the shock and the space variable is now taken to be one dimensional, $x \in \boldsymbol{R}^1$. In that approach, the Boltzmann shock profiles are approximated by the Navier-Stokes shock profiles. The approach does not yield the positivity of the shocks due to the polynomial perturbation to the exponentially small Gaussian tail in the momentum variable ξ. Our idea is to show the positivity using the time-asymptotic stability analysis. The positivity property is necessary for the profile to be physical, both for the boundedness of the entropy density $f \log f$ and for f to be the distribution function. We take as initial data a positive approximation of the exact profile with the same total macroscopic variables, Sect. 6. In Sect. 7 we present the local macroscopic and microscopic decomposition based on local Maxwellians. The decomposition is based on the Navier-Stokes profile through the Chapman-Enskog expansion. The fluid speeds are studied in Sect. 8 and the stability of shocks shown in Sect. 9. It is well-known that positive initial data for the Boltzmann equation yield positive solutions. Thus the time-asymptotic stability of the exact profile yields its non-negativity. The positivity is then shown by a maximum principle for the full Boltzmann equation satisfied by the profile, Sect. 10.

We choose the perturbation to have zero total macroscopic variables. This choice allows us to study the positivity of the shock profiles using only the energy method. For the energy method for viscous conservation laws, see [6], [11], [16]. Without the zero macroscopic variables condition, there should be interaction of fluid waves of distinct families. For the study of the interactions for viscous conservation laws using pointwise estimates see [13]. It would be interesting to study the nonlinear stability of shock profiles with general perturbation. This is, however, left to the future.

The stability analysis for the shock profile demands a sufficient accuracy of the approximate profile. The approximate profile is obtained by Chapman-Enskog expansion. Instead of using the Lyapunov-Schmidt process as in [3], we apply the weighted energy estimate method to macro-micro equations for the construction of the shock profiles. The method requires the detailed study of the tail behavior of the approximate profiles. For this, we make good uses of the exact expression of the dissipation parameters and collision frequency ,[5]. The analysis is carried out in the Appendix for the hard spheres and yields the needed stronger accuracy estimate.

2. Macro-Micro Decomposition

Consider the Boltzmann equation

$$F_t + \xi \cdot \nabla_x F = Q(F, F), \tag{2.1}$$

and a Maxwellian state

$$\omega(\xi) \equiv \frac{e^{-\frac{|\xi - u_0|^2}{2T_0}}}{\sqrt{(2\pi T_0)^3}} \text{ for } \xi \in \boldsymbol{R}^3.$$

The collision invariants $\chi_i(\xi), i = 0, \cdots, 4$, are normalized with respect to the Maxwellian state:

$$\int_{\boldsymbol{R}^3} \chi_i \; Q(g,h)\; d\xi = 0 \text{ for } i = 0, \cdots 4, \tag{2.2}$$

$$\begin{cases} \chi_0(\xi) \equiv 1, \\ \chi_i(\xi) \equiv \dfrac{\xi^i - u_0^i}{\sqrt{T_0}} \text{ for } i = 1,2,3, \\ \chi_4(\xi) \equiv \dfrac{1}{\sqrt{6}} \left(\dfrac{|\xi - u_0|^2}{T_0} - 3 \right), \end{cases}$$

$$\langle \omega^{\frac{1}{2}} \chi_i, \omega^{\frac{1}{2}} \chi_j \rangle = \delta_{ij}.$$

Here $\langle , \rangle$ is the inner product of the Hilbert space, $\mathbb{H} \equiv L^2(\boldsymbol{R}^3)$, for the momentum variable ξ:

$$\langle g, h \rangle \equiv \int_{\boldsymbol{R}^3} g \; h \; d\xi \text{ for } g, h \in \mathbb{H}.$$

The Maxwellian ω satisfies $Q(\omega, \omega) = 0$, and the Boltzmann solution around ω satisfies

$$F(x,t,\xi) \equiv \omega(\xi; u_0, T_0) + \omega^{\frac{1}{2}}(\xi; u_0, T_0) \; f(x,t,\xi) \text{ for } x = (x^1, x^2, x^3) \in \boldsymbol{R}^3,$$

$$\begin{aligned} & f_t + \xi \cdot \nabla_x f - Lf = \mathcal{N}(f), \\ & L(h) \equiv \omega^{-\frac{1}{2}} \; [Q(\omega, \omega^{\frac{1}{2}} h) + Q(\omega^{\frac{1}{2}} h, \omega)], \\ & \mathcal{N}(f) \equiv \omega^{-\frac{1}{2}} Q(\omega^{\frac{1}{2}} f, \omega^{\frac{1}{2}} f). \end{aligned} \tag{2.3}$$

The linearized collision operator L is self-adjoint, non-positive, and has the same collision invariants as Q, [7]:

$$\langle Lg, h \rangle = \langle g, Lh \rangle, \; \langle Lg, g \rangle \leq 0, \tag{2.4}$$

$$L(\omega^{\frac{1}{2}} \chi_i) = 0, \; i = 0, \cdots, 4. \tag{2.5}$$

In fact, $\omega^{\frac{1}{2}} \chi_i, \; i = 0, \cdots, 4$ span the null space of L. We denote by $\boldsymbol{P}_0$ the projection on this null space and $\boldsymbol{P}_1$ the complementary projection:

$$\begin{cases} f_0 \equiv \boldsymbol{P}_0 f, \\ \boldsymbol{P}_0 f \equiv \sum_{i=0}^4 \langle \omega^{\frac{1}{2}} \chi_i, f \rangle \; \omega^{\frac{1}{2}} \chi_i, \\ f_0(x,t,\xi) \equiv \Big(\rho(x,t) + \sum_{i=1}^3 m^i(x,t) \; \chi_i(\xi) + e(x,t) \; \chi_4(\xi) \Big) \; \omega^{\frac{1}{2}}, \\ f_1 = \boldsymbol{P}_1 f, \; \boldsymbol{P}_1 f \equiv f - f_0, \\ \mathbb{P}_0 \equiv Range(\boldsymbol{P}_0), \\ \mathbb{P}_1 \equiv \mathbb{P}_0^{\perp}, \\ \rho(x,t) \equiv \Big\langle \omega^{\frac{1}{2}} \chi_0, f \Big\rangle, \\ m^i(x,t) \equiv \Big\langle \omega^{\frac{1}{2}} \chi_i, f \Big\rangle \text{ for } i = 1,2,3, \\ e(x,t) \equiv \Big\langle \omega^{\frac{1}{2}} \chi_4, f \Big\rangle. \end{cases}$$

It follows that

$$L\boldsymbol{P}_0 f = Lf_0 = 0,\ Lf = L\boldsymbol{P}_1 f = Lf_1.$$

Here, we introduce new norms for $f(x, t, \xi)$:

$$\begin{cases} \|f\|_{L^2_\xi}(x, t) \equiv \langle f, f\rangle^{\frac{1}{2}}, \\ \|f\|_{L^\infty_{x,t}(L^2_\xi)} \equiv \sup_{(x,t)\in \boldsymbol{R}\times\boldsymbol{R}^+} \|f\|_{L^2_\xi}(x, t). \end{cases}$$

Remark 2.1. The variables f_0 and f_1, respectively, are called the *macroscopic* and *microscopic* components of f. The macroscopic component is often also called the fluid component; and the microscopic component the non-fluid component. The Hilbert space $\mathbb{H}$ is decomposed into the macroscopic space $\mathbb{P}_0 \equiv ker(L)$ and its orthogonal complement $\mathbb{P}_1$. Thus $f_0 \in \mathbb{P}_0$ and $f_1 \in \mathbb{P}_1$. The decomposition is standard. The main interest here is the decomposition of the Boltzmann equation and the energy estimates to be followed.

For hard sphere, [4], and Grad's cutoff potentials, [7], L is the sum of a multiplication operator $-\nu(\xi)$, the collision frequency, and a compact operator K:

$$Lh(\xi) = -\nu(\xi)h(\xi) + K(h)(\xi).$$

The function $\nu(\xi)$ has a positive lower bound. These facts, (2.4) and (2.5) imply that there exists $\nu_0 > 0$ such that any $h \in \mathbb{P}_1$,

$$\langle h, Lh\rangle \leq -\nu_0\langle h, h\rangle. \tag{2.6}$$

We will write the negative operator in the space $\mathbb{P}_1$ as

$$\bar{L} \equiv L|_{\mathbb{P}_1} \leq -\nu_0.$$

The linearized Boltzmann equation

$$f_t + \xi \cdot \nabla_x f = Lf \tag{2.7}$$

is decomposed into macroscopic and microscopic equations:

$$\begin{aligned} &f_{0t} + \boldsymbol{P}_0\xi \cdot \nabla_x f_0 = -\boldsymbol{P}_0\xi \cdot \nabla_x f_1, \\ &f_{1t} + \boldsymbol{P}_1\xi \cdot \nabla_x f_1 - Lf_1 = -\boldsymbol{P}_1\xi \cdot \nabla_x f_0, \end{aligned} \tag{2.8}$$

It will be useful later to have from the second equation the following expression for the function f_1:

$$f_1 = \bar{L}^{-1}\left(f_{1t} + \boldsymbol{P}_1\xi \cdot \nabla_x(f_0 + f_1)\right),$$

and rewrite the first equation as

$$f_{0t} + \boldsymbol{P}_0\xi \cdot \nabla_x f_0 + \boldsymbol{P}_0\xi \cdot \nabla_x \bar{L}^{-1}\left(f_{1t} + \boldsymbol{P}_1\xi \cdot \nabla_x(f_0 + f_1)\right) = 0. \tag{2.9}$$

Similarly, the full Boltzmann equation (2.3) is decomposed as:

$$\begin{cases} f_{0t} + \boldsymbol{P}_0\xi \cdot \nabla_x(f_0 + f_1) = 0, \\ f_{1t} + \boldsymbol{P}_1\xi \cdot \nabla_x(f_0 + f_1) = (1 + \rho)Lf_1 + \boldsymbol{N}(f). \end{cases} \tag{2.10}$$

The nonlinear term $\boldsymbol{N}(f)$ is expressed as follows:

$$\begin{aligned} \mathscr{N}(f) &\equiv \omega^{-\frac{1}{2}} Q(\rho\,\omega + (\boldsymbol{h} + f_1)\omega^{\frac{1}{2}}, \rho\,\omega + (\boldsymbol{h} + f_1)\omega^{\frac{1}{2}}) \\ &= \rho(x, t)\, Lf_1 + \boldsymbol{N}(f), \\ \boldsymbol{N}(f) &\equiv \omega^{-\frac{1}{2}} Q(\omega^{\frac{1}{2}}(\boldsymbol{h} + f_1), \omega^{\frac{1}{2}}(\boldsymbol{h} + f_1)), \end{aligned} \tag{2.11}$$

where the perturbation has been written as $f = (\rho(x, t) + \boldsymbol{h}(x, t, \xi))\omega^{\frac{1}{2}} + f_1(x, t, \xi)$.

3. Representation of the Macroscopic Variables

The macroscopic dissipation, for both Boltzmann and Navier-Stokes equations, is for momentum and energy, but not for mass. In other words, it applies to the following function $\boldsymbol{h}(x,t,\xi)$:

$$\boldsymbol{h}(x,t,\xi) \equiv f_0 - \rho(x,t)\,\omega^{\frac{1}{2}} = \left(\sum_{i=1}^{3} m^i\ \psi_i + e\ \psi_4\right)\omega^{\frac{1}{2}}.$$

The following lemma describes an expression of dissipation resulting from the energy estimate in the next section from the macroscopic equation, cf. (4.2), (4.3),

Lemma 3.1. *For any macroscopic function* $f_0 \equiv (\rho(x,t)+m^i(x,t)\chi_i(\xi)+e(x,t)\chi_4(\xi))\,\omega^{\frac{1}{2}}(\xi)$ *and* $\eta \in \boldsymbol{R}^3$*, there exists* $C > 0$ *such that*

$$\begin{aligned}
&\langle \boldsymbol{P}_1\eta\cdot\xi f_0, \boldsymbol{P}_1\eta\cdot\xi f_0\rangle(x,t)\\
&\quad\le \left(\|\eta\|^2\cdot\|\boldsymbol{m}\|^2 + \frac{(\eta\cdot\boldsymbol{m})^2+\sum_{i=1}^3(m^i\eta^i)^2}{3} + 4\|\eta\|^2e^2\right)(x,t); \qquad (3.1)\\
&C^{-1}\left(\|\eta\|^2\cdot\|\boldsymbol{m}\|^2 + \frac{(\eta\cdot\boldsymbol{m})^2+\sum_{i=1}^3(m^i\eta^i)^2}{3} + 4\|\eta\|^2e^2\right)\\
&\quad\le \left|\langle \bar{L}^{-1}\boldsymbol{P}_1\eta\cdot\xi f_0, \bar{L}^{-1}\boldsymbol{P}_1\eta\cdot\xi f_0\rangle\right|\\
&\quad\le C\left(\|\eta\|^2\cdot\|\boldsymbol{m}\|^2 + \frac{(\eta\cdot\boldsymbol{m})^2+\sum_{i=1}^3(m^i\eta^i)^2}{3} + 4\|\eta\|^2e^2\right); \qquad (3.2)\\
&C^{-1}\sum_{i=1}^{3}\langle\partial_{x^i}\boldsymbol{h},\partial_{x^i}\boldsymbol{h}\rangle \le \langle\boldsymbol{P}_1\xi\cdot\nabla_x f_0, \boldsymbol{P}_1\xi\cdot\nabla_x f_0\rangle \le C\sum_{i=1}^{3}\langle\partial_{x^i}\boldsymbol{h},\partial_{x^i}\boldsymbol{h}\rangle;\\
&\int_{\boldsymbol{R}^3}\langle\boldsymbol{P}_1\xi\cdot\nabla_x f_0, \boldsymbol{P}_1\xi\cdot\nabla_x f_0\rangle\,dx \qquad (3.3)\\
&\quad= \int_{\boldsymbol{R}^3}\sum_{i=1}^{3}\frac{(\partial_{x^i}m^i)^2}{3} + \frac{(div\cdot\boldsymbol{m})^2}{3} + \sum_{1\le i,j\le 3}(\partial_{x^i}m^j)^2 + 4\sum_{i=1}^{3}(\partial_{x^i}e)^2dx,
\end{aligned}$$

where $\|\cdot\|$ *is a standard Euclidean norm in* $\boldsymbol{R}^3$.

Proof. We have, by straightforward computations,

$$\begin{aligned}
&\boldsymbol{P}_1\ \xi^i\chi_0\omega^{1/2} = 0,\\
&\boldsymbol{P}_1\ \xi^i\chi_j\omega^{1/2} = \xi^i\chi_j - \tfrac{1}{3}\delta^{ij}\textstyle\sum_{k=1}^3(\xi^k)^2 \text{ for } 1\le i,j\le 3,\\
&\boldsymbol{P}_1\ \xi^i\chi_4\omega^{1/2} = \xi^i\chi_4 - \tfrac{2}{\sqrt{6}}\xi^i \text{ for } 1\le i\le 3,
\end{aligned}$$

$\langle \boldsymbol{P}_1\, \xi^i \chi_j \omega^{\frac{1}{2}}, \boldsymbol{P}_1\, \xi^{i'} \chi_{j'} \omega^{\frac{1}{2}} \rangle$	$1 \le i, i', j, j' \le 3$
1	$(i \neq j)$ & $(i' \neq j')$ & $(i = i')$& $(j = j')$
1	$(i \neq j)$ & $(i' \neq j')$ & $(i = j')$& $(j = i')$
0	$(i \neq j)$ & $(i' \neq j')$ & $\{i, j\} \neq \{i', j'\}$
0	$(i \neq j)$ & $(i' = j')$
0	$(i = j)$ & $(i' \neq j')$
$-\frac{2}{3}$	$(i = j)$ & $(i' = j')$&$(i \neq i')$
$\frac{4}{3}$	$(i = j)$ & $(i' = j')$&$(i = i')$

$$\langle \boldsymbol{P}_1\, \xi^i \chi_4 \omega^{\frac{1}{2}}, \boldsymbol{P}_1\, \xi^{i'} \chi_{j'} \omega^{\frac{1}{2}} \rangle = 0,$$
$$\langle \boldsymbol{P}_1\, \xi^i \chi_4 \omega^{\frac{1}{2}}, \boldsymbol{P}_1\, \xi^{i'} \chi_4 \omega^{\frac{1}{2}} \rangle = 4\delta^{ii'}.$$

The above table yields

$$\sum_{\substack{1\le i,i'\le 3\\ 1\le j,j'\le 4}} \eta_i \eta_{i'} b^j b^{j'} \langle \boldsymbol{P}_1 \xi^i \chi_j \omega^{\frac{1}{2}}, \boldsymbol{P}_1 \xi^{i'} \chi_{j'} \omega^{\frac{1}{2}} \rangle$$
$$= \sum_{i=1}^{3} (\eta_j)^2 \sum_{j=1}^{3} (b^j)^2 + \frac{1}{3} \sum_{i=1}^{3} (\eta_j)^2 (b^j)^2 + \left(\sum_{i=1}^{3} \eta_i b_i \right)^2 + 4 \sum_{i=1}^{3} (\eta_i)^2 (b_4)^2. \quad (3.4)$$

This equality yields (3.1) as well as (3.3).

Consider the following diagram:

$$\begin{array}{ccccc} \mathbb{P}_0 & \xrightarrow{\boldsymbol{P}_1 \xi \cdot \eta} & \mathbb{P}_1 & \xrightarrow{\bar{L}^{-1}} & \mathbb{P}_1 \\ & & \uparrow & & \uparrow \\ & & \boldsymbol{P}_1 \xi \cdot \eta(\mathbb{P}_0) & \xrightarrow{\bar{L}^{-1}|_{\boldsymbol{P}_1 \xi \cdot \eta(\mathbb{P}_0)}} & \bar{L}^{-1}(\boldsymbol{P}_1 \xi \cdot \eta(\mathbb{P}_0)). \end{array}$$

Since the operator $L : \mathbb{P}_1 \longmapsto \mathbb{P}_1$ is injective and the dimension of $\boldsymbol{P}_1 \xi \cdot \eta(\mathbb{P}_0)$ is finite, the operator $\bar{L}^{-1}|_{\boldsymbol{P}_1 \xi \cdot \eta(\mathbb{P}_0)}$ is bijective, bounded and invertible. Thus there exists $C > 0$ such that

$$\frac{1}{C} \langle \boldsymbol{P}_1 \xi \cdot \eta f_0, \boldsymbol{P}_1 \xi \cdot \eta f_0 \rangle \le \langle \bar{L}^{-1}(\boldsymbol{P}_1 \xi \cdot \eta f_0), \bar{L}^{-1}(\boldsymbol{P}_1 \xi \cdot \eta f_0) \rangle$$
$$\le C \langle \boldsymbol{P}_1 \xi \cdot \eta f_0, \boldsymbol{P}_1 \xi \cdot \eta f_0 \rangle$$

for all $\|\eta\| = 1$. This proves (3.2). $\square$

The proof of the next lemma is straightforward and is omitted.

Lemma 3.2. *There exists $C > 0$ such that the following holds for all $\|\eta\| = 1$ and $\mathbb{P}_1$-valued L^2-function f_1:*

$$\int_{\boldsymbol{R}^3} \langle \boldsymbol{P}_0 \xi \cdot \eta f_1, \boldsymbol{P}_0 \xi \cdot \eta f_1 \rangle dx \le C \int_{\boldsymbol{R}^3} \langle f_1, f_1 \rangle dx.$$

4. Energy Estimates for Linear Equation

The preliminary energy estimate is obtained directly from (2.7), making use of (2.6):

$$\begin{aligned}
0 &= \int_0^\tau \int_{\boldsymbol{R}^3} \left\langle f_{x^i}, \partial_{x^i}(f_t + \xi \cdot \nabla_x f - Lf) \right\rangle dxdt \\
&= \int_0^\tau \int_{\boldsymbol{R}^3} \frac{1}{2}\partial_t \left\langle f_{x^i}, f_{x^i} \right\rangle + \left\langle f_{x^i}, \xi \cdot \nabla_x f_{x^i} \right\rangle - \left\langle f_{x^i}, Lf_{x^i} \right\rangle dxdt \\
&= \frac{1}{2} \int_{\boldsymbol{R}^3} \left\langle f_{x^i}, f_{x^i} \right\rangle dx \Big|_{t=0}^{t=\tau} - \int_0^\tau \int_{\boldsymbol{R}^3} \left\langle f_{x^i}, Lf_{x^i} \right\rangle dxdt \\
&\geq \frac{1}{2} \int_{\boldsymbol{R}^3} \left\langle f_{x^i}, f_{x^i} \right\rangle dx \Big|_{t=0}^{t=\tau} + \nu_0 \int_0^\tau \int_{\boldsymbol{R}^3} \left\langle f_{1x^i}, f_{1x^i} \right\rangle dxdt
\end{aligned}$$

for $i = 1, 2, 3$.

This yields

$$\frac{1}{2} \int_{\boldsymbol{R}^3} \langle f_{x^i}, f_{x^i} \rangle dx \Big|_{t=0}^{t=\tau} + \nu_0 \int_0^t \int_{\boldsymbol{R}^3} \langle f_{1x^i}, f_{1x^i} \rangle \, dxdt \leq 0 \text{ for } i = 1, 2, 3. \tag{4.1}$$

Note that we have obtained only partial dissipation, that is, only the microscopic dissipation f_{1x}.

Next we perform the energy estimates for the macroscopic component by integrating f_0 times (2.9):

$$\begin{aligned}
&\frac{1}{2} \int_{\boldsymbol{R}^3} \langle f_0, f_0 \rangle (x,t) dx \Big|_{t=0}^{t=\tau} - \sum_{i=1}^{3} \int_0^\tau \int_{\boldsymbol{R}^3} \langle f_{0x^i}, \boldsymbol{P}_0 \xi^i \bar{L}^{-1} \boldsymbol{P}_1 \xi \cdot \nabla_x f_0 \rangle (x,t) dxdt \\
&\qquad + \int_0^\tau \int_{\boldsymbol{R}^3} \langle f_0, \boldsymbol{P}_0 \xi \cdot \nabla_x \bar{L}^{-1} (f_{1t} + \boldsymbol{P}_1 \xi \cdot \nabla_x f_1) \rangle (x,t) dxdt = 0.
\end{aligned} \tag{4.2}$$

From (2.6) and Lemma 3.1, there exist C and $C_1 > 0$ such that

$$\begin{aligned}
&-\sum_{i=1}^{3} \int_0^\tau \int_{\boldsymbol{R}^3} \langle f_{0x^i}, \boldsymbol{P}_0 \xi^i \bar{L}^{-1} \boldsymbol{P}_1 \xi \cdot \nabla_x f_0 \rangle (x,t) dxdt \\
&= -\int_0^\tau \int_{\boldsymbol{R}^3} \langle (\bar{L} \cdot \bar{L}^{-1}) \boldsymbol{P}_1 \xi \cdot \nabla_x \bar{f}_0, \bar{L}^{-1} \boldsymbol{P}_1 \xi \cdot \nabla_x f_0 \rangle (x,t) dxdt \\
&\geq \nu_0 \int_0^\tau \int_{\boldsymbol{R}^3} \langle \bar{L}^{-1} \boldsymbol{P}_1 \xi \cdot \nabla_x f_0, \bar{L}^{-1} \boldsymbol{P}_1 \xi \cdot \nabla_x f_0 \rangle (x,t) dxdt \\
&> C_1 \nu_0 \int_0^\tau \int_{\boldsymbol{R}^3} \langle \boldsymbol{P}_1 \xi \cdot \nabla_x f_0, \boldsymbol{P}_1 \xi \cdot \nabla_x f_0 \rangle (x,t) dxdt \\
&> C \nu_0 \sum_{i=0}^{3} \int_0^\tau \int_{\boldsymbol{R}^3} \langle \partial_{x^i} \boldsymbol{h}, \partial_{x^i} \boldsymbol{h} \rangle (x,t) dxdt.
\end{aligned} \tag{4.3}$$

Combine (4.2) and (4.3) to yield

$$\frac{1}{2}\int_{\boldsymbol{R}^3}\langle f_0, f_0\rangle(x,t)dx\Big|_{t=0}^{t=\tau}+\nu_0 C\sum_{i=1}^{3}\int_0^\tau\int_{\boldsymbol{R}^3}\langle\partial_{x^i}\boldsymbol{h},\partial_{x^i}\boldsymbol{h}\rangle(x,t)dxdt$$
$$+\int_0^\tau\int_{\boldsymbol{R}^3}\langle f_0, \boldsymbol{P}_0\xi\cdot\nabla_x\bar{L}^{-1}(f_{1t}+\boldsymbol{P}_1\xi\cdot\nabla_x f_1)\rangle(x,t)dxdt\le 0. \tag{4.4}$$

By the Schwartz inequality and (3.3), there exist $C,\ C_1>0$ such that

$$\int_0^\tau\int_{\boldsymbol{R}^3}\langle f_0, \boldsymbol{P}_0\xi\cdot\nabla_x\bar{L}^{-1}f_{1t}\rangle dxdt=-\int_0^t\int_{\boldsymbol{R}^3}\langle\boldsymbol{P}_1\xi\cdot\nabla_x f_0, \bar{L}^{-1}f_{1t}\rangle dxdt$$
$$\le\frac{\nu_0 C}{4}\sum_{i=1}^{3}\int_0^\tau\int_{\boldsymbol{R}^3}\langle\partial_{x^i}\boldsymbol{h},\partial_{x^i}\boldsymbol{h}\rangle dxdt+C_1\int_0^\tau\int_{\boldsymbol{R}^3}\langle f_{1t}, f_{1t}\rangle dxdt;$$

$$\int_0^\tau\int_{\boldsymbol{R}^3}\langle f_0, \xi\cdot\nabla_x\bar{L}^{-1}\xi\cdot\nabla_x f_1\rangle dxdt=\sum_{j=1}^{3}\int_0^\tau\int_{\boldsymbol{R}^3}\langle\boldsymbol{P}_1\xi^j\bar{L}^{-1}\boldsymbol{P}_1\xi\cdot\nabla_x f_0, f_{1x^j}\rangle dxdt$$
$$\le\sum_{i=1}^{3}\left(\frac{C\nu_0}{8}\int_0^\tau\int_{\boldsymbol{R}^3}\langle\partial_{x^i}\boldsymbol{h},\partial_{x^i}\boldsymbol{h}\rangle dxdt+C_1\int_0^\tau\int_{\boldsymbol{R}^3}\langle f_{1x^i}, f_{1x^i}\rangle dxdt\right);$$

and

$$\int_0^\tau\int_{\boldsymbol{R}^3}\langle\boldsymbol{P}_0\xi^i\partial_{x^i}f_1, \boldsymbol{P}_0\xi^j\partial_{x^j}\bar{L}^{-1}f_1\rangle dxdt\le C_1\sum_{i=1}^{3}\int_0^\tau\int_{\boldsymbol{R}^3}\langle f_{1x^i}, f_{1x^i}\rangle dxdt.$$

This, (4.4) and (4.5) yield the macroscopic dissipation:

$$\frac{1}{4}\int_{\boldsymbol{R}^3}\langle f_0, f_0\rangle dx\Big|_{t=0}^{t=\tau}+\frac{C_1\nu_0}{2}\int_0^t\int_{\boldsymbol{R}^3}\langle\boldsymbol{P}_1\xi\cdot\nabla_x f_0, \boldsymbol{P}_1\xi\cdot\nabla_x f_0\rangle dxdt$$
$$\le 3C_0\left(\int_0^t\int_{\boldsymbol{R}^3}\langle f_{1t}, f_{1t}\rangle dxdt+\sum_{i=1}^{3}\int_0^\tau\int_{\boldsymbol{R}^3}\langle f_{1x^i}, f_{1x^i}\rangle dxdt\right). \tag{4.6}$$

From (4.6) and (4.1), we can choose $\gamma>0$ sufficiently small to result in

$$\int_{\boldsymbol{R}^3}\left(\langle f_0, f_0\rangle+\gamma^{-1}\sum_{i=0}^{3}\langle f_{x^i}, f_{x^i}\rangle\right)(x,t)dx\Big|_{t=0}^{t=\tau}$$
$$+\frac{\nu_0 C}{4}\sum_{i=1}^{3}\int_0^\tau\int_{\boldsymbol{R}^3}\langle\partial_{x^i}\boldsymbol{h},\partial_{x^i}\boldsymbol{h}\rangle\, dxdt+\gamma^{-1}\nu_0\sum_{i=1}^{3}\int_0^\tau\int_{\boldsymbol{R}^3}\langle f_{1x^i}, f_{1x^i}\rangle dxdt<0. \tag{4.7}$$

This basic energy estimate yields the stability of the global Maxwellian states for the linearized Boltzmann equation.

5. Nonlinear Stability of Planar Waves

In preparation for later study of the positivity of shock profiles, we will only consider the planar wave propagation, $x \in \boldsymbol{R}^1$,

$$\begin{gathered} F_t + \xi^1 F_x = Q(F, F); \\ F(x, t, \xi) \in \boldsymbol{R}, \quad (x, t, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^+ \times \boldsymbol{R}^3, \end{gathered} \tag{5.1}$$

and assume that the perturbation has zero total macroscopic variables:

$$\int_{\boldsymbol{R}} \langle \chi_i \ \omega^{\frac{1}{2}}, f \rangle \ dx \bigg|_{t=0} = 0 \text{ for } i = 0, \cdots, 4, \ \ i.e. \int_{\boldsymbol{R}} f_0 dx = 0.$$

The macroscopic component plays a different role from the microscopic component and the above zero mass assumption simplifies the analysis, cf. Remark 5.2. From (2.2),

$$\begin{gathered} 0 = \int_{\boldsymbol{R}} \int_{\boldsymbol{R}^3} \chi_i \ (F_t + \xi^1 F_x - Q(F, F)) \ d\xi dx = \int_{\boldsymbol{R}} \int_{\boldsymbol{R}^3} \chi_i F_t \ d\xi dx; \\ \int_{\boldsymbol{R}} \langle \chi_i \ \omega^{\frac{1}{2}}, f \rangle \ dx \bigg|_t = \int_{\boldsymbol{R}} \langle \chi_i \ \omega^{\frac{1}{2}}, f \rangle \ dx \bigg|_0 = 0 \text{ for } i = 0, \cdots, 4 \text{ and } t \geq 0. \end{gathered}$$

This yields

$$\int_{\boldsymbol{R}} f_0 \ dx \bigg|_t \equiv 0 \text{ for all } t \geq 0,$$

and prompts us to introduce the anti-derivative of the macroscopic component:

$$W_0(x, t, \xi) \equiv \int_{-\infty}^{x} f_0(y, t, \xi) \ dy.$$

Integrate the first equation in (2.10) to obtain

$$W_{0t} + \boldsymbol{P}_0 \xi^1 W_{0x} + \boldsymbol{P}_0 \xi^1 f_1 = 0, \tag{5.2a}$$
$$f_{1t} + \boldsymbol{P}_1 \xi^1 (f_{1x} + W_{0xx}) = L f_1 + \mathscr{N}(f, f) = (1 + \rho) \ L f_1 + \boldsymbol{N}(f), \tag{5.2b}$$
$$W_{0x} = f_0.$$

Write

$$W_0(x, t, \xi) \equiv \{R(x, t) \ \chi_0(\xi) + M(x, t) \ \chi_1(\xi) + E(x, t) \ \chi_4(\xi)\} \, \omega^{\frac{1}{2}}.$$

$$\begin{cases} R_x = \rho, \\ M_x = m, \\ E_x = e; \end{cases} \tag{5.3}$$

and from (5.2a),

$$\begin{cases} R_t + M_x = 0, \\ M_t + R_x + \dfrac{2}{\sqrt{6}} E_x + \langle \chi_1 \omega^{\frac{1}{2}}, \xi^1 f_1 \rangle = 0, \\ E_t + \dfrac{2}{\sqrt{6}} M_x + \langle \chi_4 \omega^{\frac{1}{2}}, \xi^1 f_1 \rangle = 0. \end{cases}$$

Denote the sup norm for W_0:

$$|||W_0|||_\infty \equiv \sup_{(x,t)\in \boldsymbol{R}\times\boldsymbol{R}^+} (|R| + |M| + |E|)\,(x,t).$$

For the variables $W_0(x,t)$, we make an *a priori* smallness assumption:

$$\|(1+|\xi|)^{\frac{1}{2}} f_1\|_{L^\infty_{x,t}(L^2_\xi)} + |||W_0|||_\infty \ll 1. \tag{5.4}$$

For these new variables (R, M, E), we have that

$$\begin{aligned}\int_0^\tau\int_{\boldsymbol{R}} R_x M_t dxdt &= \int_{\boldsymbol{R}} R_x M dx\Big|_{t=0}^{t=\tau} + \int_0^\tau\int_{\boldsymbol{R}} R_t M_x dxdt\\ &= \int_{\boldsymbol{R}} R_x M dx\Big|_0^\tau - \int_{t=0}^{t=\tau}\int_{\boldsymbol{R}} M_x M_x dxdt.\end{aligned}$$

This results in

$$\begin{aligned}0 &= \int_0^\tau\int_{\boldsymbol{R}} R_x\Big(M_t + R_x + \frac{2}{\sqrt{6}}E_x + \Big\langle \chi_1\omega^{\frac{1}{2}}, \xi^1 f_1\Big\rangle dxdt\\ &= \int_{\boldsymbol{R}} R_x M dx\Big|_{t=0}^{t=\tau} + \int_0^\tau\int_{\boldsymbol{R}} -|M_x|^2 + |R_x|^2 + R_xE_x + R_x\Big\langle \chi_1\omega^{\frac{1}{2}}, \xi^1 f_1\Big\rangle dxdt.\end{aligned}$$

By the Schwartz inequality, one can conclude

$$\int_0^\tau\int_{\boldsymbol{R}} \rho^2 dxdt = O(1)\left|\left(\int_{\boldsymbol{R}} \rho M dx\Big|_{t=0}^{t=\tau}\right)\right| + O(1)\int_0^\tau\int_{\boldsymbol{R}} m^2 + e^2 + \langle f_1, f_1\rangle\, dxdt. \tag{5.5}$$

Remark 5.1. The procedure for the estimate $\int_0^\tau\int_{\boldsymbol{R}}\rho^2 dxdt$ is motivated by the method in [9].

Lemma 5.2. *For any* $i \geq 0$,

$$\begin{cases}\|\partial_x^i \boldsymbol{h}\|^2_{L^2_\xi} = O(1)(|\partial_x^i m|^2 + |\partial_x^i e|^2),\\ \|\partial_x^i\partial_t \boldsymbol{h}\|^2_{L^2_\xi} = O(1)(|\partial_x^i\partial_t m|^2 + |\partial_x^i\partial_t e|^2).\end{cases}$$

For the nonlinear term $\boldsymbol{N}(f)$, we have the following lemma:

Lemma 5.3. *For any $i \geq 0$ it follows*

$$\begin{aligned}
\left\langle \partial_x^i f_1, \partial_x^i \boldsymbol{N}(f) \right\rangle \leq O(1) & \left(\sum_{0 \leq \beta \leq \left[\frac{i}{2}\right]} \|(1+|\xi|)^{\frac{1}{2}} \partial_x^{i-\beta} f_1\|_{L_\xi^2}^2 + |\partial_x^{i-\beta} m|^2 + |\partial_x^{i-\beta} e|^2 \right) \\
\cdot & \left(\sum_{0 \leq \beta \leq \left[\frac{i}{2}\right]} \|\partial_x^\beta f_1\|_{L_{x,t}^\infty(L_\xi^2)} + \|\partial_x^\beta m\|_\infty + \|\partial_x^\beta e\|_\infty \right) \\
& + O(1) \left(\sum_{0 \leq \beta \leq \left[\frac{i}{2}\right]} \|\partial_x^{i-\beta} f_1\|_{L_\xi^2}^2 + |\partial_x^{i-\beta} m|^2 + |\partial_x^{i-\beta} e|^2 \right) \\
& \cdot \left(\sum_{0 \leq \beta \leq \left[\frac{i}{2}\right]} \|(1+|\xi|)^{\frac{1}{2}} \partial_x^\beta f_1\|_{L_{x,t}^\infty(L_\xi^2)} + \|\partial_x^\beta m\|_\infty + \|\partial_x^\beta e\|_\infty \right); \qquad (5.6)
\end{aligned}$$

$$\begin{aligned}
& \left\langle \partial_x^i \partial_t f_1, \partial_x^i \partial_t \boldsymbol{N}(f) \right\rangle \qquad (5.7) \\
& \leq O(1) \left(\sum_{0 \leq \beta \leq \left[\frac{i}{2}\right]} \|(1+|\xi|)^{\frac{1}{2}} \partial_x^{i-\beta} \partial_t f_1\|_{L_\xi^2}^2 + |\partial_x^{i-\beta} \partial_t m|^2 + |\partial_x^{i-\beta} \partial_t e|^2 \right. \\
& \quad \left. + \|(1+|\xi|)^{\frac{1}{2}} \partial_x^{i-\beta} f_1\|_{L_\xi^2}^2 + |\partial_x^{i-\beta} m|^2 + |\partial_x^{i-\beta} e|^2 \right) \\
& \quad \cdot \left(\sum_{0 \leq \beta \leq \left[\frac{i}{2}\right]} \|\partial_x^\beta \partial_t f_1\|_{L_{x,t}^\infty(L_\xi^2)} + \|\partial_x^\beta \partial_t m\|_\infty + \|\partial_x^\beta \partial_t e\|_\infty \right. \\
& \quad \left. + \|\partial_x^\beta f_1\|_{L_{x,t}^\infty(L_\xi^2)} + \|\partial_x^\beta m\|_\infty + \|\partial_x^\beta e\|_\infty \right) \\
& \quad + O(1) \left(\sum_{0 \leq \beta \leq \left[\frac{i}{2}\right]} \|\partial_x^{i-\beta} \partial_t f_1\|_{L_\xi^2}^2 + |\partial_x^{i-\beta} \partial_t m|^2 + |\partial_x^{i-\beta} \partial_t e|^2 \right. \\
& \quad \left. + \|\partial_x^{i-\beta} f_1\|_{L_\xi^2}^2 + |\partial_x^{i-\beta} m|^2 + |\partial_x^{i-\beta} e|^2 \right)
\end{aligned}$$

$$\cdot\left(\sum_{0\le\beta\le\left[\frac{i}{2}\right]}\|(1+|\xi|)^{\frac{1}{2}}\partial_x^\beta\partial_t f_1\|_{L^\infty_{x,t}(L^2_\xi)}+\|\partial_x^\beta\partial_t m\|_\infty+\|\partial_x^\beta\partial_t e\|_\infty\right.$$
$$\left.+\sum_{0\le\beta\le\left[\frac{i}{2}\right]}\|(1+|\xi|)^{\frac{1}{2}}\partial_x^\beta f_1\|_{L^\infty_{x,t}(L^2_\xi)}+\|\partial_x^\beta m\|_\infty+\|\partial_x^\beta e\|_\infty\right).$$

This lemma is a consequence of Lemma 5.2, (2.11), Lemma B.1, and the following

$$\langle f_1,\boldsymbol{N}(f)\rangle=\left\langle(1+|\xi|)^{\frac{1}{2}}f_1,(1+|\xi|)^{-\frac{1}{2}}\boldsymbol{N}(f)\right\rangle.$$

5.1. Lower Order Energy Estimates. Similar to (2.9), we have

$$W_{0t}+\boldsymbol{P}_0\xi^1W_{0x}+\ \boldsymbol{P}_0\xi^1\bar{L}^{-1}(\boldsymbol{P}_1\xi^1W_{0xx}+f_{1t}+\boldsymbol{P}_1\xi f_{1x}-\rho Lf_1-\boldsymbol{N}(f))=0.$$

Multiply the above equation by W_0 and integrate it over $[0,\tau]\times\boldsymbol{R}$:

$$\int_0^\tau\int_{-\infty}^\infty\Big\langle W_0,W_{0t}+\boldsymbol{P}_0\xi^1W_{0x}$$
$$+\boldsymbol{P}_0\xi^1\bar{L}^{-1}(\boldsymbol{P}_1\xi^1W_{0xx}+f_{1t}+\boldsymbol{P}_1\xi f_{1x}-\rho Lf_1-\boldsymbol{N}(f))\Big\rangle dxdt=0.\quad(5.8)$$

Since (5.2a) differs from (2.8) in the differentiation order of the variable f_1, we need to modify the procedure for (4.7) in order to obtain the nonlinear stability for the variable W_0.

By (3.1) in Lemma 3.1, there exists $C>0$ such that

$$C\,\langle\boldsymbol{h},\boldsymbol{h}\rangle\le\left\langle\boldsymbol{P}_0\xi^1\boldsymbol{P}_1\xi^1W_{0x},\boldsymbol{P}_0\xi^1\boldsymbol{P}_1\xi^1W_{0x}\right\rangle.$$

Thus, similar to (4.2), we have

$$\frac{1}{2}\int_{\boldsymbol{R}}\langle W_0,W_0\rangle dx\Big|_{t=0}^{t=\tau}+C\nu_0\int_0^\tau\int_{\boldsymbol{R}}\langle\boldsymbol{h},\boldsymbol{h}\rangle\,dxdt+\int_0^\tau\int_{\boldsymbol{R}}\langle\boldsymbol{P}_1\xi^1W_0,\bar{L}^{-1}f_{1t}\rangle\,dxdt$$
$$+\int_0^\tau\int_{\boldsymbol{R}}\langle\boldsymbol{P}_1\xi^1W_0,\bar{L}^{-1}\boldsymbol{P}_1\xi f_{1x}\rangle\,dxdt$$
$$-\int_0^\tau\int_{\boldsymbol{R}}\langle\boldsymbol{P}_1\xi^1W_0,\rho f_1\rangle\,dxdt$$
$$-\int_0^\tau\int_{\boldsymbol{R}}\langle\boldsymbol{P}_1\xi^1W_0,\bar{L}^{-1}\boldsymbol{N}(f)\rangle\,dxdt\le 0.\quad(5.9)$$

The second double integral in (5.9) can be estimated as follows: For any $\gamma_1 > 0$,

$$\begin{aligned}
&\int_0^\tau \int_{\boldsymbol{R}} \langle \boldsymbol{P}_1 \xi^1 W_0, \bar{L}^{-1} f_{1t} \rangle dxdt - \int_{\boldsymbol{R}} \langle \boldsymbol{P}_1 \xi^1 W_0, \bar{L}^{-1} f_1 \rangle dx \Big|_{t=0}^{t=\tau} \\
&\quad = -\int_0^\tau \int_{\boldsymbol{R}} \langle \boldsymbol{P}_1 \xi^1 W_{0t}, \bar{L}^{-1} f_1 \rangle dxdt \\
&\quad = \int_0^\tau \int_{\boldsymbol{R}} \langle \boldsymbol{P}_1 \xi^1 \boldsymbol{P}_0 (\xi^1 W_{0x} + \xi^1 f_1), \bar{L}^{-1} f_1 \rangle dxdt \\
&\quad \le \int_0^\tau \int_{\boldsymbol{R}} \left| \left\langle \boldsymbol{P}_1 \xi^1 \boldsymbol{P}_0 \xi^1 W_{0x}, \bar{L}^{-1} f_1 \right\rangle \right| dxdt + O(1) \int_0^\tau \int_{\boldsymbol{R}} \langle f_1, f_1 \rangle dxdt \\
&\quad \le C\gamma_1 \int_0^\tau \int_{\boldsymbol{R}} \langle \boldsymbol{h}, \boldsymbol{h} \rangle + \rho^2 dxdt + O(1)(1 + \gamma_1^{-1}) \int_0^\tau \int_{\boldsymbol{R}} \langle f_1, f_1 \rangle dxdt. \qquad (5.10)
\end{aligned}$$

By integration by parts and Schwartz's inequality, there exists $C > 0$ such that

$$\begin{aligned}
&\int_0^\tau \int_{\boldsymbol{R}} \langle \boldsymbol{P}_1 \xi^1 W_0, \bar{L}^{-1} \boldsymbol{P}_1 \xi f_{1x} \rangle \, dxdt \\
&\quad = -\int_0^\tau \int_{-\infty}^{\infty} \left\langle \boldsymbol{P}_1 \xi^1 \bar{L}^{-1} \boldsymbol{P}_1 \xi^1 W_{0x}, f_1 \right\rangle dxdt \\
&\quad \le 2 \int_0^\tau \int_{\boldsymbol{R}} \gamma_1 \, \langle \boldsymbol{P}_1 \xi^1 \bar{L}^{-1} \boldsymbol{P}_1 \xi^1 W_{0x}, \boldsymbol{P}_1 \xi^1 \bar{L}^{-1} \boldsymbol{P}_1 \xi^1 W_{0x} \rangle + \frac{1}{\gamma_1} \langle f_1, f_1 \rangle \, dxdt \\
&\quad \le 2C \int_0^\tau \int_{\boldsymbol{R}} \gamma_1 \, \langle \boldsymbol{h}, \boldsymbol{h} \rangle + \frac{1}{\gamma_1} \langle f_1, f_1 \rangle \, dxdt \text{ for all } \gamma_1 > 0. \qquad (5.11)
\end{aligned}$$

From the structure of $\boldsymbol{N}(f)$ in (2.11) and Lemma B.1 we have from the Schwartz inequality

$$\|(1 + |\xi|)^{-\frac{1}{2}} \boldsymbol{N}(f)\|_{L^2_\xi} \le O(1) \left(m^2 + e^2 + \|(1 + |\xi|)^{\frac{1}{2}} f_1\|^2_{L^2_\xi} \right).$$

From this, there exists C

$$\begin{aligned}
&\int_0^\tau \int_{\boldsymbol{R}} \langle \boldsymbol{P}_1 \xi^1 W_0, \bar{L}^{-1} \boldsymbol{N}(f) \rangle dxdt \\
&\quad = \int_0^\tau \int_{\boldsymbol{R}} \langle (1 + |\xi|)^{\frac{1}{2}} \boldsymbol{P}_1 \xi^1 W_0, (1 + |\xi|)^{-\frac{1}{2}} \bar{L}^{-1} \boldsymbol{N}(f) \rangle dxdt \\
&\quad \le C |||W_0|||_\infty \int_0^\tau \int_{\boldsymbol{R}} m^2 + e^2 + \|(1 + |\xi|)^{\frac{1}{2}} f_1\|^2_{L^2_\xi} \, dxdt. \qquad (5.12)
\end{aligned}$$

By the Schwartz inequality and (5.5),

$$\begin{aligned}
&\int_0^\tau \int_{\boldsymbol{R}} \langle \boldsymbol{P}_1 \xi^1 W_0, \rho f_1 \rangle dx dt \\
&\quad \le \frac{1}{2} \| \boldsymbol{P}_1 \xi^1 W_0 \|_{L^\infty_{x,t}(L^2_\xi)} \int_0^\tau \int_{\boldsymbol{R}} \rho^2 + \langle f_1, f_1 \rangle \, dx dt \\
&\quad \le O(1) |||W_0|||_\infty \left(\int_{\boldsymbol{R}} \rho M dx \Big|_{t=0}^{t=\tau} + \int_0^\tau \int_{\boldsymbol{R}} m^2 + e^2 + \|(1+|\xi|)^{\frac{1}{2}} f_1\|^2_{L^2_\xi} dx dt \right).
\end{aligned} \tag{5.13}$$

From the above, for a small $\gamma_1 > 0$ satisfying

$$2\gamma_1 C < \frac{\nu_0}{4},$$

$$\begin{aligned}
&\int_{\boldsymbol{R}} \frac{1}{2} \langle W_0, W_0 \rangle + \langle \boldsymbol{P}_1 \xi^1 W_0, \bar{L}^{-1} f_1 \rangle dx \Big|_{t=0}^{t=\tau} + \frac{\nu_0}{8} \int_0^\tau \int_{\boldsymbol{R}} \langle \boldsymbol{h}, \boldsymbol{h} \rangle dx dt \\
&\quad \le O(1) \int_0^\tau \int_{-\infty}^\infty \langle f_1, f_1 \rangle \, dx dt + O(1) |||W_0|||_\infty \left| \left(\int_{\boldsymbol{R}} \rho^2 + M^2 dx \Big|_{t=0}^{t=\tau} \right) \right|.
\end{aligned} \tag{5.14}$$

By Schwartz's inequality, there exists C such that for all $\gamma_1 > 0$,

$$\begin{aligned}
\int_R \left\langle \boldsymbol{P}_1 \xi^1 W_0, \bar{L}^{-1} f_1 \right\rangle dx &\le \int_R \gamma_1 \left\langle \boldsymbol{P}_1 \xi^1 W_0, \boldsymbol{P}_1 \xi^1 W_0 \right\rangle + \frac{1}{\gamma_1} \left\langle \bar{L}^{-1} f_1, \bar{L}^{-1} f_1 \right\rangle dx \\
&\le C \int_R \gamma_1 \langle W_0, W_0 \rangle + \frac{1}{\gamma_1} \langle f_1, f_1 \rangle \, dx.
\end{aligned} \tag{5.15}$$

Combine (5.14) and (5.15) with the estimate in (5.5) to bound the term $\int_0^t \int_R \rho^2 dx dt$ in (5.10). Then, under the smallness condition (5.4), it follows that there exists $\gamma_1 \in (0, 1)$ such that

$$\begin{aligned}
&\int_{\boldsymbol{R}} \frac{1}{4} \langle W_0, W_0 \rangle + \frac{C}{\gamma_1} \langle f_1, f_1 \rangle \, dx \Big|_{t=\tau} + \frac{\nu_0}{8} \int_0^\tau \int_{\boldsymbol{R}} \langle \boldsymbol{h}, \boldsymbol{h} \rangle dx dt \\
&\quad \le \int_{\boldsymbol{R}} \frac{1}{2} \langle W_0, W_0 \rangle + \frac{2C}{\gamma_1} \langle f_1, f_1 \rangle + O(1) |||W_0|||_\infty \, \rho^2 dx \Big|_{t=0} \\
&\qquad + O(1) \, |||W_0|||_\infty \int_{\boldsymbol{R}} \rho^2 dx \Big|_{t=\tau} \\
&\qquad + O(1) \int_0^\tau \int_{\boldsymbol{R}} \|(1+|\xi|)^{\frac{1}{2}} f_1\|^2_{L^2_\xi} dx dt.
\end{aligned} \tag{5.16}$$

Similar to (4.1), we consider the following energy estimate using the original Boltzmann equation (2.10) to handle $O(1) \int_0^\tau \int_{-\infty}^\infty \langle f_1, f_1 \rangle \, dx dt$ in the RHS of (5.16):

$$\int_0^t \int_{-\infty}^\infty \left\langle f, f_t + \xi^1 f_x - (1+\rho) L f - \boldsymbol{N}(f) \right\rangle dx dt = 0.$$

This and Lemma 5.3 results in the following energy estimate:

$$\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty}\langle f, f\rangle\,dx\Big|_{t=0}^{t=\tau}-\int_0^{\tau}\int_{-\infty}^{\infty}(1+\rho)\ \langle f_1, Lf_1\rangle\,dxdt\\
&=\int_0^{\tau}\int_{-\infty}^{\infty}\langle f_1, \boldsymbol{N}(f)\rangle\,dxdt\\
&\le O(1)\int_0^{\tau}\int_{-\infty}^{\infty}\|(1+|\xi|)^{\frac{1}{2}}f_1\|_{L^{\infty}_{x,t}(L^2_\xi)}(m^2+e^2)+\|f_1\|_{L^{\infty}_{x,t}(L^2_\xi)}\|(1+|\xi|)^{\frac{1}{2}}f_1\|^2_{L^2_\xi}dxdt.
\end{aligned}\tag{5.17}$$

Under the smallness assumption (5.4), that is, for some $\gamma\in(0,1)$,

$$\|(1+|\xi|)^{\frac{1}{2}}f_1\|_{L^{\infty}_{x,t}(L^2_\xi)}+|||W_0|||_{\infty}\ll\gamma\ll\gamma_1<1,$$

the combination of (5.16) + $\frac{2}{\gamma}$(5.17) results in, for some $C>1$,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\langle W_0, W_0\rangle+\frac{1}{2C\gamma}\langle f, f\rangle\,dx\Big|_{t=\tau}\\
&\quad+\frac{\nu_0}{8C}\int_0^{\tau}\int_{\boldsymbol{R}}\langle \boldsymbol{h}, \boldsymbol{h}\rangle+\frac{1}{\gamma}\|(1+|\xi|)^{\frac{1}{2}}f_1\|^2_{L^2_\xi}dxdt\\
&\le\int_{-\infty}^{\infty}2\,\langle W_0, W_0\rangle+\frac{1}{C\gamma}\langle f, f\rangle\,dx\Big|_{t=0}.
\end{aligned}\tag{5.18}$$

This concludes the lower order energy estimates for $W_0(x,t)$ under the smallness assumption (5.4).

Remark 5.4. We have derived this lower energy estimate as a consequence of the following combination:

$$\begin{aligned}
&\int_0^{\tau}\int_{-\infty}^{\infty}\Big\langle W_0, W_{0t}+\xi^1W_{0x}+\boldsymbol{P}_0\xi^1f_1\Big\rangle dxdt\\
&\quad+\int_0^{\tau}\int_{-\infty}^{\infty}\Big\langle W_0, \bar{L}^{-1}(f_{1t}+\boldsymbol{P}_1\xi^1f_{1x}-\rho\bar{L}f_1-\boldsymbol{N}(f))\Big\rangle dxdt\\
&\quad+\frac{1}{\gamma}\int_0^{\tau}\int_{-\infty}^{\infty}\Big\langle f, f_t+\xi^1f_x-\bar{L}f-\rho\bar{L}f-\boldsymbol{N}(f)\Big\rangle dxdt=0.
\end{aligned}\tag{5.19}$$

□

Remark 5.5. In the lower order energy estimate (5.18), the integral $\int_{\boldsymbol{R}}\frac{1}{2C\gamma}\langle f, f\rangle\,dx\Big|_{t=\tau}$ does not contain a component $\Big\langle(1+|\xi|)^{\frac{1}{2}}f_1, (1+|\xi|)^{\frac{1}{2}}f_1\Big\rangle$, which will be used to close the smallness assumption (5.4). Instead of using the time boundary integral, we use the double integral $\int_0^{\tau}\int_{\boldsymbol{R}}\langle(1+|\xi|)f_1, f_1\rangle\,dxdt$ and $\int_0^{\tau}\int_{\boldsymbol{R}}\langle(1+|\xi|)\partial_tf_1, \partial_tf_1\rangle\,dxdt$ to resolve the smallness assumption.

5.2. Higher Order Energy Estimates. In order to complete the nonlinear stability analysis, we need to extend the lower energy estimates to the high order derivatives of $W_0(x,t,\xi)$. For this, we need to estimate:

$$O(1)\ \Big(\|W_0\|_{H_x^6(L_\xi^2)} + \|(1+|\xi|)^{\frac{1}{2}} f_1\|_{H_x^6(L_\xi^2)} \\ +\|\partial_t W_0\|_{H_x^6(L_\xi^2)} + \|(1+|\xi|)^{\frac{1}{2}}\partial_t f_1\|_{H_x^6(L_\xi^2)}\Big),$$

here $\|\cdot\|_{H_x^\beta(L_\xi^2)}$ is the Sobolev norm,

$$\|g\|^2_{H_x^\beta(L_\xi^2)} \equiv \sum_{0\le|\alpha|\le\beta}\int_{\boldsymbol{R}}\left\langle \partial_x^\alpha g, \partial_x^\alpha g\right\rangle dx.$$

The order of the Sobolev norm is chosen to guarantee the following smallness *a priori* assumption:

$$\sum_{|\alpha|\le 4}\left(|||\partial_x^\alpha W_0|||_\infty + |||\partial_x^\alpha \partial_t W_0|||_\infty\right) + \sum_{|\alpha|\le 3}\left(\|\partial_x^\alpha f_1\|_{L^\infty_{x,t}(L_\xi^2)} + \|\partial_x^\alpha\partial_t f_1\|_{L^\infty_{x,t}(L_\xi^2)}\right) \\ \le \varsigma_0 \text{ for some given } \varsigma_0 \ll 1, \tag{5.20}$$

needed to take care of the nonlinear term. This assumption is consistent with the smallness condition (5.4).

Lemma 5.6. *For any g_i satisfying $\boldsymbol{P}_0 g_i \equiv 0$,*

$$|\left\langle g_1, Lg_2\right\rangle| \le -\frac{1}{2}\left\{\delta\left\langle g_1, Lg_1\right\rangle + \delta^{-1}\left\langle g_2, Lg_2\right\rangle\right\},$$

for any constant $\delta > 0$.

Proof. Since both g_1 and g_2 are in $\mathbb{P}_1$,

$$\sqrt{\delta}g_1 + \frac{1}{\sqrt{\delta}}g_2,\ \sqrt{\delta}g_1 - \frac{1}{\sqrt{\delta}}g_2 \in \mathbb{P}_1.$$

Thus, we have

$$\left\langle\sqrt{\delta}g_1 + \frac{1}{\sqrt{\delta}}g_2, L\left(\sqrt{\delta}g_1 + \frac{1}{\sqrt{\delta}}g_2\right)\right\rangle \le 0, \left\langle\sqrt{\delta}g_1 - \frac{1}{\sqrt{\delta}}g_2, L\left(\sqrt{\delta}g_1 - \frac{1}{\sqrt{\delta}}g_2\right)\right\rangle \le 0.$$

This and the self-adjoint property of L ,(2.4), result in the lemma. □

Let γ_0 be a given positive number satisfying $\varsigma \ll \gamma_0 \ll 1$. Consider the following combination:

$$
\begin{aligned}
&\sum_{0\le i\le 6} \gamma_0^{-i} \int_0^\tau \int_{\boldsymbol{R}} \Big\langle \partial_x^i W_0, \partial_x^i (W_{0t} + \boldsymbol{P}_0\xi^1 W_{0x} + \boldsymbol{P}_0 \xi^1 f_1 \Big\rangle dxdt \\
&\quad + \sum_{0\le i\le 6} \gamma_0^{-i} \int_0^\tau \int_{\boldsymbol{R}} \Big\langle \partial_x^i W_0, \partial_x^i \boldsymbol{P}_0 \xi^1 \bar{L}^{-1}((f_{1t} + \boldsymbol{P}_1 \xi^1 f_{1x} \\
&\quad + \boldsymbol{P}_1 \xi^1 f_{0x} - \rho \bar{L} f_1 - \boldsymbol{N}(f))) \Big\rangle dxdt \\
&\quad + \sum_{0\le i\le 6} \gamma_0^{-i-1} \int_0^\tau \int_{\boldsymbol{R}} \Big\langle \partial_x^i f, \partial_x^i (f_t + \xi f_x - Lf - \mathscr{N}(f)) \Big\rangle dxdt \\
&\quad + \sum_{0\le i\le 6} \gamma_0^{-i} \int_0^\tau \int_{\boldsymbol{R}} \Big\langle \partial_x^i \partial_t W_0, \partial_x^i \partial_t (W_{0t} + \boldsymbol{P}_0\xi^1 W_{0x} + \boldsymbol{P}_0 \xi^1 f_1 \Big\rangle dxdt \\
&\quad + \sum_{0\le i\le 6} \gamma_0^{-i} \int_0^\tau \int_{\boldsymbol{R}} \Big\langle \partial_x^i \partial_t W_0, \partial_x^i \partial_t \boldsymbol{P}_0 \xi^1 \bar{L}^{-1}((f_{1t} + \boldsymbol{P}_1 \xi^1 f_{1x} \\
&\quad + \boldsymbol{P}_1 \xi^1 f_{0x} - \rho \bar{L} f_1 - \boldsymbol{N}(f))) \Big\rangle dxdt \\
&\quad + \sum_{0\le i\le 6} \gamma_0^{-i-1} \int_0^\tau \int_{\boldsymbol{R}} \Big\langle \partial_x^i \partial_t f, \partial_x^i \partial_t (f_t + \xi f_x - Lf - \mathscr{N}(f)) \Big\rangle dxdt = 0. \qquad (5.21)
\end{aligned}
$$

This generalizes (5.19), which was used to obtain the lower order energy estimate (5.18).

Finally,

$$
\begin{aligned}
&\sum_{i=0}^{6} \gamma_0^{-i} \int_{-\infty}^{\infty} \|\partial_x^i W_0\|_{L_\xi^2}^2 + \|\partial_x^i \partial_t W_0\|_{L_\xi^2}^2 + \frac{\|\partial_x^i f\|_{L_\xi^2}^2 + \|\partial_x^i \partial_t f\|_{L_\xi^2}^2}{2C\gamma} dx \Bigg|_{t=\tau} \\
&+ \sum_{i=0}^{6} \gamma_0^{-i} \frac{\nu_0}{8C} \int_0^\tau \int_{\boldsymbol{R}} \|\partial_x^i \boldsymbol{h}\|_{L_\xi^2}^2 + \|\partial_x^i \partial_t \boldsymbol{h}\|_{L_\xi^2}^2 dxdt \\
&+ \sum_{i=0}^{6} \gamma_0^{-i} \frac{\nu_0}{8C\gamma} \int_0^\tau \int_{\boldsymbol{R}} \|(1+|\xi|)^{\frac{1}{2}} \partial_x^i f_1\|_{L_\xi^2}^2 + \|(1+|\xi|)^{\frac{1}{2}} \partial_x^i \partial_t f_1\|_{L_\xi^2}^2 dxdt \\
&\le \sum_{i=0}^{6} \gamma_0^{-i} \int_{-\infty}^{\infty} 2\left(\|\partial_x^i W_0\|_{L_\xi^2}^2 + \|\partial_x^i \partial_t W_0\|_{L_\xi^2}^2 \right) + \frac{1}{C\gamma} \left(\|\partial_x^i f\|_{L_\xi^2}^2 + \|\partial_x^i \partial_t f\|_{L_\xi^2}^2 \right) dx \Bigg|_{t=0}. \qquad (5.22)
\end{aligned}
$$

By the Sobolev inequality with (5.22) and the following inequality:

$$
\begin{aligned}
&\int_{\boldsymbol{R}} \|(1+|\xi|)^{\frac{1}{2}} \partial_x^i f_1\|_{L^2_\xi}^2 dx \Big|_{t=\tau} \\
&\quad \le \int_{\boldsymbol{R}} \|(1+|\xi|)^{\frac{1}{2}} \partial_x^i f_1\|_{L^2_\xi}^2 dx \Big|_{t=0} \\
&\quad\quad + \int_0^\tau \int_{\boldsymbol{R}} \|(1+|\xi|)^{\frac{1}{2}} \partial_x^i f_1\|_{L^2_\xi} \|(1+|\xi|)^{\frac{1}{2}} \partial_x^i \partial_t f_1\|_{L^2_\xi} dx dt \\
&\quad \le 2 \int_{\boldsymbol{R}} \|(1+|\xi|)^{\frac{1}{2}} \partial_x^i f_1\|_{L^2_\xi}^2 dx \Big|_{t=0} + \int_0^\tau \int_{\boldsymbol{R}} \|(1+|\xi|)^{\frac{1}{2}} \partial_x^i f_1\|_{L^2_\xi}^2 \\
&\quad\quad + \|(1+|\xi|)^{\frac{1}{2}} \partial_x^i \partial_t f_1\|_{L^2_\xi}^2 dx dt
\end{aligned} \tag{5.23}
$$

the smallness assumption (5.20) is justified when the initial data is sufficiently small compared to ς_0. Thus, (5.22) concludes the nonlinear stability.

6. Shock Profiles

Let $\phi(x-st,\xi)$ be a travelling wave solution of the Boltzmann equation (5.1)

$$
-s\phi' + \xi^1 \phi' = Q(\phi,\phi). \tag{6.1}
$$

Denote by $\phi_{\mathscr{T}}(x-st,\xi)$ and $\omega_{\mathscr{T}}(x-st,\xi)$ the corresponding profiles of local thermal equilibrium distributions:

$$
\begin{cases}
\phi_{\mathscr{T}}(x-st) \equiv \rho(x-st) \dfrac{e^{-\frac{|\xi-u|^2}{2T}}}{\sqrt{(2\pi T)^3}}; \\
\omega_{\mathscr{T}}(x-st,\xi) \equiv \dfrac{e^{-\frac{|\xi-u|^2}{2T}}}{\sqrt{(2\pi T)^3}},
\end{cases} \tag{6.2}
$$

where (ρ, u, T) are the macroscopic variables of the travelling wave solution:

$$
\begin{cases}
\rho(x-st) \equiv \displaystyle\int_{\boldsymbol{R}^3} \phi(x-st,\xi)\, d\xi, \\
m(x-st) \equiv \displaystyle\int_{\boldsymbol{R}^3} \xi^1 \phi(x-st,\xi)\, d\xi, \\
E(x-st) \equiv \displaystyle\int_{\boldsymbol{R}^3} \frac{|\xi|^2}{2} \phi(x-st,\xi)\, d\xi, \\
\left(\dfrac{m^2}{2\rho} + \rho\, T\right) \equiv E \equiv \rho \left(\dfrac{u^2}{2} + e\right).
\end{cases}
$$

The states $(\rho_\pm, m_\pm, E_\pm) \equiv \lim_{x\to\pm\infty}(\rho, m, E)(x)$ satisfy the Rankine-Hugoniot condition:

$$
\begin{cases}
s(\rho_- - \rho_+) = m_- - m_+, \\
s(m_- - m_+) = (u_- m_- + p_-) - (u_+ m_+ + p_+), \\
s(E_- - E_+) = (u_-(E_- + p_-) - u_+(E_+ + p_+)).
\end{cases} \tag{R-H}
$$

We consider a weak shock with the following normalization:

$$\begin{cases} R = 1 \text{ (Gas Constant)} \\ \rho_- = 1, \\ u_- = 0, \\ T_- = 1, \\ p_- - p_+ > 0, \text{(Entropy Condition)} \\ |\rho_- - \rho_+| \ll 1, \\ \epsilon \equiv |\rho_- - \rho_+|, \\ \left| s - \frac{\sqrt{5}}{\sqrt{3}} \right| \ll 1. \end{cases} \tag{6.3}$$

Let $(\rho_{NS}, u_{NS}, E_{NS})(x - st)$ be the Navier-Stokes shock profile (A.9) obtained through the Chapman-Enskog expansion. This profile connects the same end states $(\rho_\pm, u_\pm, E_\pm)$. We denote the corresponding local Maxwellians by

$$\begin{cases} \phi_{tr}(x - st, \xi) \equiv \rho_{NS} \dfrac{e^{-\frac{|\xi - u_{NS}|^2}{2T_{NS}}}}{\sqrt{(2\pi T_{NS})^3}} \\ \omega_{tr}(x - st, \xi) \equiv \dfrac{e^{-\frac{|\xi - u_{NS}|^2}{2T_{NS}}}}{\sqrt{(2\pi T_{NS})^3}}. \end{cases} \tag{6.4}$$

7. Local Macroscopic and Microscopic Variables

Denote by $J(x, t; \xi)$ the perturbation of the travelling wave $\phi(x - st, \xi)$ with local Maxwellian initial state:

$$\begin{cases} F(x, t, \xi) \equiv \phi(x - st, \xi) + J(x, t, \xi), \\ F(x, 0, \xi) \equiv \phi_{\mathscr{T}}(x, \xi). \end{cases} \tag{7.1}$$

The equation for $J(x, t, \xi)$ is

$$\begin{cases} \partial_t J + \xi^1 J_x = Q(\phi + J, \phi + J) - Q(\phi, \phi), \\ J(x, 0, \xi) \equiv \phi_{\mathscr{T}}(x, \xi) - \phi(x, \xi). \end{cases} \tag{7.2}$$

With the change of coordinates:

$$\begin{cases} x \to x - st, \\ t \to t, \end{cases}$$

the shock profile becomes stationary, and (5.1) and (7.2) become

$$\begin{aligned} &F_t + (\xi^1 - s)F_x = Q(F, F), \\ &J_t + (\xi^1 - s)J_x = Q(J + \phi, J + \phi) - Q(\phi, \phi). \end{aligned} \tag{7.3}$$

Since a shock profile is orbitally stable, to properly locate the shock, we use the conservation laws for the macroscopic variables. Our choice of the initial state yields

$$\int_{-\infty}^{\infty}\int_{\boldsymbol{R}^3} J(x,0,\xi)\, d\xi dx = \int_{-\infty}^{\infty}\int_{\boldsymbol{R}^3} J(x,t,\xi)\, d\xi dx = 0,$$
$$\int_{-\infty}^{\infty}\int_{\boldsymbol{R}^3} \xi^i J(x,0,\xi)\, d\xi dx = \int_{-\infty}^{\infty}\int_{\boldsymbol{R}^3} \xi^i J(x,t,\xi)\, d\xi dx = 0 \text{ for } i = 1,\cdots,3,$$
$$\int_{-\infty}^{\infty}\int_{\boldsymbol{R}^3} |\xi|^2 J(x,0,\xi)\, d\xi dx = \int_{-\infty}^{\infty}\int_{\boldsymbol{R}^3} |\xi|^2 J(x,t,\xi)\, d\xi dx = 0.$$

This implies that the perturbation does not cause the time asymptotic phase shift of the shock, [12]. Consider the anti-derivative:

$$W(x,t,\xi) \equiv \int_{-\infty}^{x} J(y,t,\xi)\, dy;$$

$$W_t + (\xi^1 - s)W_x - \int_{-\infty}^{x} Q(J+\phi, J+\phi) - Q(\phi,\phi) dy = 0. \tag{7.4}$$

We use the local Maxwellian profiles $\phi_{tr}(x,\xi)$ and $\omega_{tr}(x,\xi)$ for the macro-micro decomposition. Let L and $\mathscr{L}$, respectively, be the linearized collision operator around ϕ_{tr} and ϕ:

$$\begin{cases} LJ \equiv Q(\phi_{tr}, J) + Q(J, \phi_{tr}), \\ \mathscr{L}J \equiv Q(\phi, J) + Q(J, \phi). \end{cases}$$

We also introduce the following deviations:

$$\begin{cases} DJ \equiv \mathscr{L}J - LJ, \\ \mathscr{N}(J) \equiv Q(J+\phi, J+\phi) - Q(\phi,\phi) - LJ - DJ. \end{cases}$$

The functions $\psi_i(x,\xi)\omega_{tr}(\xi),\ 1 = 0,\cdots,4$, span the kernel of L:

$$L(\psi_i \omega_{tr}) = 0 \text{ for } i = 0,\cdots,4,$$

$$\begin{cases} \psi_0(x,\xi) \equiv\ 1, \\ \psi_i(x,\xi) \equiv \dfrac{(\xi^i - u^i_{NS})}{\sqrt{T_{NS}}} \text{ for } i = 1,2,3 \\ \psi_4(x,\xi) \equiv \dfrac{1}{\sqrt{6}}\left(\dfrac{|\xi - u_{NS}|^2}{T_{NS}} - 3\right). \end{cases}$$

Remark 7.1. These invariants $\psi_i\omega_{tr}$ are orthogonal with respect to the weight ω_{tr} in (6.2).

For $h(x,t,\xi)$ and $j(x,t,\xi)$ functions on $\boldsymbol{R} \times \boldsymbol{R}^+ \times \boldsymbol{R}^3$; and $\boldsymbol{A}$ an operator on $L^2(\boldsymbol{R} \times \boldsymbol{R}^+ \times \boldsymbol{R}^3)$, we set

$$\begin{cases} \langle h|g\rangle\,(x,t) \equiv \int_{\boldsymbol{R}^3} hg/\omega_{tr}\, d\xi, \\ \langle h|\boldsymbol{A}|g\rangle\,(x,t) \equiv \int_{\boldsymbol{R}^3} h\ Ag\ /\omega_{tr}\, d\xi, \\ (h,j)(x,t) \equiv \int_{\boldsymbol{R}^3} h\ j\ d\xi. \end{cases}$$

Remark 7.2. The inner product $\langle , \rangle$ of Sect. 2 differs from the present one. The is so because the perturbations are expressed differently in (1.2) and (7.1). The above collision invariants are orthogonal: For $0 \le i, j \le 4$,

$$\left\langle \psi_i \omega_{tr} | \psi_j \omega_{tr} \right\rangle = \delta^i_j.$$

The following lemma follows from the definition (6.2) of $\phi_{\mathscr{T}}$ by direct calculations.

Lemma 7.3. *The profile $\phi_{\mathscr{T}}$ in (6.2) satisfies*

$$\int_{\boldsymbol{R}^3} \phi - \phi_{\mathscr{T}} d\xi \equiv 0,$$

$$\int_{\boldsymbol{R}^3} \frac{\xi^i - u^i}{\sqrt{T}} (\phi - \phi_{\mathscr{T}}) d\xi \equiv 0 \textit{ for } x \in \boldsymbol{R} \textit{ and } i = 0, \cdots, 3,$$

$$\frac{1}{\sqrt{6}} \int_{\boldsymbol{R}^3} \left(\frac{|\xi - u|^2}{T} - 3 \right) (\phi - \phi_{\mathscr{T}}) d\xi = 0.$$

The macroscopic and microscopic variables, for $W(x,t,\xi)$ and $J(x,t,\xi)$ are defined by

$$\begin{cases} W_0(x,t,\xi) \equiv \sum_{i=0}^{4} W_0^i\ \psi_i \omega_{tr};\ \ J_0(x,t,\xi) \equiv \sum_{i=0}^{4} J_0^i\ \psi_i \omega_{tr}, \\ W_0^i \equiv \langle W|\psi_i \omega_{tr}\rangle\,,\ \ J_0^i \equiv \langle J|\psi_i \omega_{tr}\rangle\,, \\ \boldsymbol{P}_0 W \equiv W_0;\ \ \boldsymbol{P}_0 J \equiv J_0, \end{cases}$$
$$\begin{cases} W_1(x,t,\xi) \equiv W(x,t,\xi) - W_0(x,t,\xi);\ \ J_1(x,t,\xi) \equiv J(x,t,\xi) - J_0(x,t,\xi), \\ \boldsymbol{P}_1 W \equiv W_1;\ \ \boldsymbol{P}_1 J \equiv J_1. \end{cases}$$

Remark 7.4. The function W_0 is the macroscopic component of the anti-derivative W, and not the anti-derivative of the macroscopic component J_0, see Lemma 7.6 and the remark following it. The two are the same when the underline Maxwellian state is global, as in Sect. 5.

Definition 7.5. *$\mathscr{M}(x,t,\xi)$ is a purely microscopic function iff $\mathscr{M}_0(x,t,\xi) \equiv 0$ for all x and t. $\mathscr{M}(x,t,\xi)$ is a purely macroscopic function iff $\mathscr{M}_1(x,t,\xi) \equiv 0$ for all x and t, where*

$$\begin{cases} \mathscr{M}_0 \equiv \sum_{i=0}^4 \langle \psi_i \omega | \mathscr{M}\rangle\ \psi_i \omega, \\ \mathscr{M}_1 \equiv \mathscr{M} - \mathscr{M}_0. \end{cases}$$

.

Lemma 7.6. *If $\mathscr{M}(x,t,\xi)$ is purely microscopic, then $\mathscr{M}_t(x,t,\xi)$ and $\mathscr{M}_x(x,t,\xi)$ are purely microscopic.*

Proof. Since $\mathscr{M}$ is purely microscopic and χ_i is independent of (x,t),

$$0=(\chi_i,\mathscr{M});\quad 0=(\chi_i,\mathscr{M})_t=(\chi_i,\mathscr{M}_t)\text{ for } i=0,\dots,4.$$

Similarly, $\mathscr{M}_x$ is also purely microscopic. □

Remark 7.7. Since the base equilibrium state ω_{tr} is local and depends on (x,t), $\mathscr{M}_t$ is not necessarily purely macroscopic even when $\mathscr{M}$ is. We have from Lemma 7.6, $\boldsymbol{P}_0W_t=W_{0t},\ \boldsymbol{P}_0W_x=W_{0x}$ and since $J=W_x$,

$$J_0=\boldsymbol{P}_0W_{0x};\ J_1=\boldsymbol{P}_1W_{0x}+W_{1x}. \tag{7.5}$$

From (7.3) and Lemma 7.6, we have the decomposition

$$\partial_t W_0+\boldsymbol{P}_0(\xi^1-s)\boldsymbol{P}_0\partial_x W_0+\boldsymbol{P}_0(\xi^1-s)J_1=0, \tag{7.6a}$$

$$J_{1t}+\boldsymbol{P}_1(\xi^1-s)(J_{1x}+J_{0x})-LJ_1=DJ+\mathscr{N}(J). \tag{7.6b}$$

From (7.6b),

$$\begin{aligned}J_1&=L^{-1}(J_{1t}+\boldsymbol{P}_1(\xi^1-s)(J_{1x}+\partial_x\boldsymbol{P}_0W_{0x}))-L^{-1}DJ-L^{-1}\mathscr{N}(J),\\ L^{-1}DJ&=L^{-1}D(\boldsymbol{P}_0\partial_xW_0+J_1).\end{aligned} \tag{7.7}$$

Substitute (7.7) into (7.6a) to result in:

$$\begin{aligned}&\partial_t W_0+\boldsymbol{P}_0(\xi^1-s)\boldsymbol{P}_0\partial_x W_0+\boldsymbol{P}_0(\xi^1-s)L^{-1}\boldsymbol{P}_1(\xi^1-s)\partial_x\boldsymbol{P}_0\partial_x W_0\\ &\quad+\boldsymbol{P}_0(\xi^1-s)L^{-1}\Big(\partial_t J_1+\boldsymbol{P}_1(\xi^1-s)\partial_x J_1-\rho LJ_1-\boldsymbol{N}(J)-DJ\Big)=0.\end{aligned} \tag{7.8}$$

Reference Macro-Micro Decomposition. Relative to the local macro-micro decomposition in the above, we introduce a reference macro-micro decomposition. Since the fluid variable (ρ,u,T) for the left states of the shock wave have been normalized in (6.3), we can use the absolute Maxwellian state $M_{[1,0,1]}(\xi)$,

$$M_{[1,0,1]}(\xi)\equiv\frac{e^{-\frac{|\xi|^2}{2}}}{\sqrt{(2\pi)^3}}$$

to define the following reference macro-micro decomposition $\boldsymbol{P}_0^{ref}+\boldsymbol{P}_1^{ref}$:

$$\begin{cases}\langle H_1,H_2\rangle_{ref}\equiv\displaystyle\int_{\boldsymbol{R}^3}\frac{H_1H_2}{M_{[1,0,1]}}d\xi,\\ \boldsymbol{P}_0^{ref}H\equiv\displaystyle\sum_{i=0}^4\big\langle\chi_iM_{[1,0,1]},H\big\rangle_{ref}\,\chi_iM_{[1,0,1]},\\ \boldsymbol{P}_1^{ref}H\equiv H-\boldsymbol{P}_0^{ref}H,\end{cases}$$

where χ_i is defined in (2.2) with $u_0^j=0$ for $j=1,2,3$ and with $T_0=1$.

Notations of Norms.

$$\begin{cases} \|H\|_{L^2_\xi} \equiv \langle H|H\rangle^{\frac{1}{2}}, \\ \|H\|_{ref,L^2_\xi} \equiv \langle H, H\rangle^{\frac{1}{2}}_{ref}, \\ \|H\|_{L^2_x(L^2_\xi)} \equiv \int_{\boldsymbol{R}} \|H\|^2_{L^2_\xi}\, dx, \\ \|H\|_{ref,L^2_x(L^2_\xi)} \equiv \int_{\boldsymbol{R}} \|H\|^2_{ref,L^2_\xi}\, dx, \\ \|H\|_{L^\infty_{x,t}(L^2_\xi)} \equiv \sup_{(x,t)\in \boldsymbol{R}\times\boldsymbol{R}^+} \|H\|_{L^2_\xi}(x,t), \\ \|H\|_{ref,L^\infty_{x,t}(L^2_\xi)} \equiv \sup_{(x,t)\in \boldsymbol{R}\times\boldsymbol{R}^+} \|H\|_{ref,L^2_\xi}(x,t). \end{cases}$$

Remark 7.8. For the macroscopic component $\boldsymbol{P}_0 H$, the two norms,$\|\boldsymbol{P}_0 H\|_{L^2_\xi}$ and $\|\boldsymbol{P}_0 H\|_{ref,L^2_\xi}$, are equivalent, that is, there exists $\mathsf{K} > 1$ such that

$$\mathsf{K}^{-1}\|\boldsymbol{P}_0 H\|_{L^2_\xi} \le \|\boldsymbol{P}_0 H\|_{ref,L^2_\xi} \le \mathsf{K}\|\boldsymbol{P}_0 H\|_{L^2_\xi}.$$

Remark 7.9. Local Existence in Time. With Lemmas B.1 and B.2, one can have the local existence of J in the norm $\|\cdot\|_{ref,L^2_x(L^2_\xi)}$ by rewriting the equation of J as follows:

$$\partial_t J + \xi^1 \partial_x J = \mathbb{L}J + \underbrace{((L-\mathbb{L})J + Q(J,J))}_{SMALL},$$

and applying the standard Picard iteration to the above equation. Here,

$$\mathbb{L}J \equiv Q(M,J) + Q(J,M).$$

About the variable $W_0 = \sum_{i=0}^4 W_0^i \phi_i \omega$, it can be represented by a finite dimensional hyperbolic system with a given source terms as follows:

$$\begin{cases} \partial_t(\chi_0, W^0) + \partial_x(\chi_0, \xi^1 W^0) + (\chi_0, \boldsymbol{P}_0\xi^1 J_1) = 0, \\ \partial_t(\chi_1, W^0) + \partial_x(\chi_1, \xi^1 W^0) + (\chi_1, \boldsymbol{P}_0\xi^1 J_1) = 0, \\ \partial_t(\chi_4, W^0) + \partial_x(\chi_4, \xi^1 W^0) + (\chi_4, \boldsymbol{P}_0\xi^1 J_1) = 0, \end{cases}$$

where $W_0^j = 0$ for $j = 2, 3$ due to the planar wave assumption.

Thus from this finite dimensional hyperbolic system, the local existence of J in $\|\cdot\|_{ref,L^2_x(L^2_\xi)}$ also results in the local existence of W_0 in $\|\cdot\|_{ref,L^2_x(L^2_\xi)}$. □

8. Basic Matrix Representations

We denote by $[\xi^1 - s]$ the matrix representation of the multiplication operator $\xi^1 - s$:

$$[\xi^1 - s]_{ij} \equiv \langle \psi_i \omega_{tr} | \xi^1 - s | \psi_j \omega_{tr}\rangle. \tag{8.1}$$

The system

$$v_t + [\xi^1 - s]v_x = 0,\ \ v \in \boldsymbol{R}^3,$$

defines a strictly hyperbolic system. We denote by λ_j, $r_j \equiv (r_j^1, r_j^2, r_j^3)^t$, and $l_j \equiv (l_j^1, l_j^2, l_j^3)$ the j^{th} eigenvalue, and the corresponding right and left eigenvectors of $[\xi^1 - s]$:

$$\begin{cases} \lambda_1 < \lambda_2 < \lambda_3, \\ \lambda_1 < \lambda_2 < 0, \\ |\lambda_3| \ll 1, \\ \left[\xi^1 - s\right] \, r_j = \lambda_j \, r_j, \\ l_j \, \left[\xi^1 - s\right] \, = \lambda_j \, l_j, \\ \sum_{k=1}^3 (r_j^k)^2 = 1, \\ \sum_{k=1}^3 l_i^k r_j^k = \delta_i^j. \end{cases} \tag{8.2}$$

The eigenvalues λ_i are

$$\begin{cases} \lambda_1 = u^1 - \sqrt{\frac{5T}{3}} - s, \\ \lambda_2 = u^1 - s, \\ \lambda_3 = u^1 + \sqrt{\frac{5T}{3}} - s. \end{cases}$$

From (8.1), the matrix $[\xi^1 - s]$ is symmetric. Combine this with the normalized condition $\sum_{i=1}^3 (r_j^i)^2 = 1$ to yield

$$\sum_{k=1}^3 l_i^k \partial_x r_i^k = 0 \text{ for } i = 1, 2, 3.$$

From the vectors r_j, we define $\boldsymbol{r}_j$ and $\boldsymbol{l}_j$ as follows:

$$\begin{cases} \boldsymbol{r}_j \equiv (r_j^1 \psi_0 + r_j^2 \psi_1 + r_j^3 \psi_4) \, \omega_{tr}, \\ \boldsymbol{l}_j \equiv (l_j^1 \psi_0 + l_j^2 \psi_1 + l_j^3 \psi_4) \, \omega_{tr}, \end{cases} \tag{8.3}$$

so that, for any $\boldsymbol{v}$,

$$\begin{cases} \boldsymbol{P}_0(\xi^1 - s)\boldsymbol{r}_j = \lambda_j \boldsymbol{r}_j, \\ \langle \boldsymbol{l}_j(\xi^1 - s) | \boldsymbol{P}_0 \boldsymbol{v} \rangle = \lambda_j \langle \boldsymbol{l}_j | \boldsymbol{v}. \rangle. \end{cases} \tag{8.4}$$

The macro-micro components satisfy, cf. (7.5),

$$\begin{cases} W_0 \equiv \sum_{j=1}^3 w_j \boldsymbol{r}_j, \\ \boldsymbol{h} \equiv J_0 - \langle \omega_{tr} | J_0 \rangle \, \omega_{tr}, \\ \boldsymbol{h} \equiv \left(h^1 \psi_1(\xi) + h^2 \psi_4(\xi)\right) \, \omega_{tr}, \\ J_0 = \boldsymbol{P}_0 \partial_x \sum_{j=1}^3 w_j \boldsymbol{r}_j, \;\; J_1 = \boldsymbol{P}_1 \partial_x \sum_{j=1}^3 w_j \boldsymbol{r}_j + \partial_x W_1. \end{cases} \tag{8.5}$$

Theorem 8.1. *There exists* $\mathscr{C}_0$*,* $\mathscr{C}_1$*,* $\mathscr{C}_2 1$*, and* $\mathscr{C}_3$*,* $\mathscr{C}_4 \in (0,1)$ *so that the profile* $\phi(x-st,\xi)$ *satisfies*

$$|\phi(x,\xi)-\phi_{tr}(x,\xi)| \le \mathscr{C}_0\, \epsilon^2\, e^{-\mathscr{C}_3\epsilon|x|}\rho \frac{e^{-\frac{|\xi-u^1(x)|^2}{2\mathscr{C}_2 T(x)}}}{\sqrt{T(x)^3}}, \tag{8.6}$$

$$|\partial_x^k\phi(x,\xi)| \le \mathscr{C}_0\, \epsilon^{1+k}\, e^{-\mathscr{C}_3\epsilon|x|}\rho \frac{e^{-\frac{|\xi-u^1(x)|^2}{2\mathscr{C}_2 T(x)}}}{\sqrt{T(x)^3}} \text{ for } k=1,\cdots,6,$$

$$\mathscr{C}_4\, \epsilon^2\, e^{-2\mathscr{C}_3\,\epsilon\,|x|} \le -\partial_x \left\langle \boldsymbol{l}_3(\xi^1-s)|\boldsymbol{r}_3\right\rangle = -\partial_x\lambda_3 \le \frac{\epsilon^2}{\mathscr{C}_4}\; e^{-\mathscr{C}_1\epsilon|x|}. \tag{8.7}$$

Remark 8.2. The above rate of the profile converging to the Maxwellian equilibrium states at $|\xi|=\infty$ improves that of [3]. The inequality (8.7) is the compressibility of sound speed across a shock profile; and (8.7) is a fact on the traveling wave solution of the compressible Navier-Stokes equations obtained by Chapman-Enskog expansion. The proof of this theorem will be given in Appendix C for the hard sphere. □

9. Nonlinear Stability of Shock Profiles

We impose an *a priori* smallness assumption:

$$\begin{aligned}&\sum_{|\alpha|\le 4}\left(\|\partial_x^\alpha W_0\|_{L^\infty_{x,t}(L^2_\xi)} + \|\partial_x^\alpha\partial_t W_0\|_{L^\infty_{x,t}(L^2_\xi)}\right)\\ &\quad + \sum_{|\alpha|\le 3}\left(\|\partial_x^\alpha J_1\|_{ref,L^\infty_{x,t}(L^2_\xi)} + \|\partial_x^\alpha\partial_t J_1\|_{ref,L^\infty_{x,t}(L^2_\xi)}\right)\\ &\le \varsigma_0 \text{ for some given } \varsigma_0 \ll 1.\end{aligned} \tag{9.1}$$

Similar to (5.8), we consider the energy estimate by integrating W_0 times (7.8):

$$\begin{aligned}&\int_0^\tau\int_{-\infty}^\infty \left\langle W_0|\partial_t W_0 + \boldsymbol{P}_0(\xi^1-s)\boldsymbol{P}_0\partial_x W_0 + \boldsymbol{P}_0(\xi^1-s)L^{-1}\boldsymbol{P}_1(\xi^1-s)\partial_x\boldsymbol{P}_0\partial_x W_0\right\rangle\\ &\qquad dxdt\\ &+\int_0^\tau\int_{-\infty}^\infty \left\langle W_0|\boldsymbol{P}_0(\xi^1-s)L^{-1}\left(\partial_t J_1 + \boldsymbol{P}_1(\xi^1-s)\partial_x J_1 - DJ - \mathscr{N}\right)\right\rangle\\ &\qquad dxdt = 0.\end{aligned} \tag{9.2}$$

This results in

$$
\begin{aligned}
&\mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3 + \mathscr{I}_4 + \mathscr{I}_5 + \mathscr{I}_6 + \mathscr{I}_7 + \mathscr{I}_8 \\
&\quad \equiv \frac{1}{2} \int_{-\infty}^{\infty} \langle W_0 | W_0 \rangle \, dx \Big|_{t=0}^{t=\tau} + \int_0^{\tau} \int_{-\infty}^{\infty} \Big\langle W_0 | \boldsymbol{P}_0 (\xi^1 - s) \boldsymbol{P}_0 \partial_x W_0 \Big\rangle dx dt \\
&\qquad + \int_0^{\tau} \int_{-\infty}^{\infty} \Big\langle W_0 | \boldsymbol{P}_0 (\xi^1 - s) L^{-1} \boldsymbol{P}_1 (\xi^1 - s) \partial_x \boldsymbol{P}_0 \partial_x W_0 \Big\rangle dx dt \\
&\qquad + \int_0^{\tau} \int_{-\infty}^{\infty} \Big\langle W_0 | \boldsymbol{P}_0 (\xi^1 - s) L^{-1} \partial_t J_1 \Big\rangle dx dt \\
&\qquad + \int_0^{\tau} \int_{-\infty}^{\infty} \Big\langle W_0 | \boldsymbol{P}_0 (\xi^1 - s) L^{-1} \boldsymbol{P}_1 (\xi^1 - s) \partial_x J_1 \Big\rangle dx dt \\
&\qquad - \int_0^{\tau} \int_{-\infty}^{\infty} \Big\langle W_0 | \boldsymbol{P}_0 (\xi^1 - s) \rho J_1 \Big\rangle dx dt \\
&\qquad - \int_0^{\tau} \int_{-\infty}^{\infty} \Big\langle W_0 | \boldsymbol{P}_0 (\xi^1 - s) L^{-1} \boldsymbol{N}(J) \Big\rangle dx dt \\
&\qquad - \int_0^{\tau} \int_{-\infty}^{\infty} \Big\langle W_0 | \boldsymbol{P}_0 (\xi^1 - s) L^{-1} D J \Big\rangle dx dt = 0.
\end{aligned}
$$

We now estimate $\mathscr{I}_i$, $i = 2, \ldots, 8$ using (8.5). First,

$$
\begin{aligned}
\mathscr{I}_2 &\equiv \int_0^{\tau} \int_{-\infty}^{\infty} \Big\langle W_0 | \boldsymbol{P}_0 (\xi^1 - s) \boldsymbol{P}_0 \partial_x W_0 \Big\rangle dx dt \\
&= \int_0^{\tau} \int_{-\infty}^{\infty} \sum_{j=1}^{3} \lambda_j \, w_j \left[w_{jx} + \sum_{k=1}^{3} w_k \left\langle \boldsymbol{l}_j | \partial_x \boldsymbol{r}_k \right\rangle \right] dx dt \\
&= \int_0^{\tau} \int_{-\infty}^{\infty} \sum_{j=1}^{3} \left(-\frac{\partial_x \lambda_j}{2} \, (w_j)^2 + \lambda_j w_j \sum_{k=1}^{3} w_k \left\langle \boldsymbol{l}_j | \partial_x \boldsymbol{r}_k \right\rangle \right) dx dt. \qquad \text{(9.3a)}
\end{aligned}
$$

Remark 9.1. In the above last double integral, the term $-\frac{\partial_x \lambda_3}{2} (w_3)^2$ is a positive term due to the compressibility of the profile (8.7). It is a good term for the energy estimates. The other two transversal terms $-\frac{\partial_x \lambda_j}{2} (w_j)^2$, $j = 1, 2$, are not necessarily positive and will be estimated later.

By (8.2) and (8.7), the term $\sum_{1 \le j,k \le 3} \lambda_j w_j w_k \left\langle \boldsymbol{l}_j | \partial_x \boldsymbol{r}_k \right\rangle$ can bounded by $O(1) |\partial_x \lambda_3| \sum_{i=1}^{3} w_i^2$. Thus there exists $C > 0$ such that $\mathscr{I}_2$ satisfies

$$
\int_0^{\tau} \int_{-\infty}^{\infty} |\partial_x \lambda_3| \left\{ (w_3)^2 - C \, \{ (w_1)^2 + (w_2)^2 \} \right\} dx dt \le \mathscr{I}_2.
$$

The term $\mathscr{I}_3$ is defined and estimated as follows:

$$\begin{aligned}\mathscr{I}_3 &\equiv \int_0^\tau \int_{-\infty}^\infty \Big\langle W_0|\boldsymbol{P}_0(\xi^1-s)L^{-1}\boldsymbol{P}_1(\xi^1-s)\partial_x\boldsymbol{P}_0\partial_x W_0\Big\rangle dxdt\\
&= \int_0^\tau \int_{-\infty}^\infty \Big\langle \boldsymbol{P}_1(\xi^1-s)W_0|L^{-1}\boldsymbol{P}_1\partial_x(\xi^1-s)J_0\Big\rangle dxdt\\
&= -\int_0^\tau \int_{-\infty}^\infty \Big\langle \boldsymbol{P}_1(\xi^1-s)J_0|L^{-1}|\boldsymbol{P}_1(\xi^1-s)J_0\Big\rangle dxdt\\
&\quad -\int_0^\tau \int_{-\infty}^\infty \Big\langle \boldsymbol{P}_1(\xi^1-s)W_0|L^{-1}|\boldsymbol{P}_{1x}(\xi^1-s)J_0\Big\rangle\\
&\quad +\Big\langle \boldsymbol{P}_1(\xi^1-s)W_0|(L^{-1})_x|\boldsymbol{P}_1(\xi^1-s)J_0\Big\rangle dxdt\\
&\quad +\int_0^\tau \int_{-\infty}^\infty \Big\langle \boldsymbol{P}_1(\xi^1-s)\boldsymbol{P}_1 W_{0x}|L^{-1}|\boldsymbol{P}_1(\xi^1-s)J_0\Big\rangle\\
&\quad +\Big\langle \boldsymbol{P}_1\frac{(\omega_{tr})_x}{\omega_{tr}}(\xi^1-s)W_0|L^{-1}|\boldsymbol{P}_1(\xi^1-s)J_0\Big\rangle dxdt. \qquad (9.3b)\end{aligned}$$

The x-derivatives in the above eventually will be applied to the profile ϕ_{tr} to generate a factor $\phi'_{tr} = O(1)\,\epsilon^2\, e^{-\mathscr{C}_3\epsilon|x|}$. This and (9.3b) yield that there exists $C > 0$ such that

$$\begin{aligned}&\left|\mathscr{I}_3 + \int_0^\tau \int_{-\infty}^\infty \Big\langle \boldsymbol{P}_1(\xi^1-s)J_0|L^{-1}|\boldsymbol{P}_1(\xi^1-s)J_0\Big\rangle dxdt\right|\\
&\le C\epsilon\left(\int_0^\tau \int_{-\infty}^\infty \epsilon^2\,\langle W_0|W_0\rangle\, e^{-2\mathscr{C}_3\epsilon|x|} + \Big\langle \boldsymbol{P}_1(\xi^1-s)J_0|\boldsymbol{P}_1(\xi^1-s)J_0\Big\rangle dxdt\right).\end{aligned}$$

Apply the estimates in (5.10), (5.11), (5.12), and (5.13) to $\mathscr{I}_4$, $\mathscr{I}_5$, $\mathscr{I}_6$, and $\mathscr{I}_7$, respectively to yield that there exists $C > 0$ such that for any $\gamma \in (0, 1)$,

$$\begin{aligned}&\Bigg|\mathscr{I}_4+\mathscr{I}_5+\mathscr{I}_6+\mathscr{I}_7 - \int_{-\infty}^\infty \Big\langle(\xi^1-s)W_0|L^{-1}J_1\Big\rangle dx\Big|_{t=0}^{t=\tau}\\
&\quad + \int_0^\tau \int_{-\infty}^\infty \Big\langle \boldsymbol{P}_1\xi^1\boldsymbol{P}_0\partial_x W_0|L^{-1}|\boldsymbol{P}_1\xi^1\boldsymbol{P}_0\partial_x W_0\Big\rangle\Bigg|\\
&\le C\int_0^\tau \int_{-\infty}^\infty \gamma\,\epsilon^2 e^{-2\mathscr{C}_3\epsilon|x|}\,\langle W_0|W_0\rangle\, dxdt\\
&\quad + \frac{1}{\gamma}\int_0^\tau \int_{\boldsymbol{R}} \|J_1\|^2_{ref,L^2_\xi} dxdt.\end{aligned}$$

Here, the first term in the RHS of the above inequality is a correction to the energy estimate in Sect. 5.1 because of the presence of the shock profile. The second term in the RHS results from the inner product $\langle Macro|Mirco\rangle$ in (5.10), (5.11), (5.12), and (5.13):

$$\begin{aligned}\langle Macro|Mirco\rangle &\le O(1)\Big\|\frac{\sqrt{M}}{\sqrt{\omega}}Macro\Big\|_{L^2_\xi}\cdot\|Mirco\|_{ref,L^2_\xi}\\
&= O(1)\|Macro\|_{L^2_\xi}\cdot\|Mirco\|_{ref,L^2_\xi} \qquad (9.3c)\end{aligned}$$

as long as $\int_{\boldsymbol{R}^3}\frac{M^3}{\omega^2}d\xi < \infty$ uniformly in x.

The remaining term $\mathscr{I}_8$ is estimated as follows:

$$\begin{aligned}\mathscr{I}_8 &\equiv -\int_0^\tau\int_{-\infty}^{\infty}\Big\langle W_0|\boldsymbol{P}_0(\xi^1-s)L^{-1}DJ\Big\rangle dxdt\\ &= -\int_0^\tau\int_{-\infty}^{\infty}\Big\langle \boldsymbol{P}_1(\xi^1-s)W_0|L^{-1}D(\boldsymbol{P}_0\partial_x W_0+J_1)\Big\rangle dxdt\\ &= -\int_0^\tau\int_{-\infty}^{\infty}\Big\langle \boldsymbol{P}_1(\xi^1-s)W_0|L^{-1}D\boldsymbol{P}_0\partial_x W_0\Big\rangle dxdt\\ &\quad -\int_0^\tau\int_{-\infty}^{\infty}\Big\langle \boldsymbol{P}_1(\xi^1-s)W_0|L^{-1}DJ_1\Big\rangle dxdt\\ &= \int_0^\tau\int_{-\infty}^{\infty} O(1)\Big\langle \boldsymbol{h}|L^{-1}DW_0\Big\rangle + O(1)|\partial\lambda_3|\ \|D\|_{ref,L^2_\xi}\ \langle W_0|W_0\rangle\, dxdt\\ &\quad -\int_0^\tau\int_{-\infty}^{\infty}\Big\langle \boldsymbol{P}_1(\xi^1-s)W_0|L^{-1}DJ_1\Big\rangle dxdt.\end{aligned}$$

From (8.6) and Lemma B.2, the operator in ξ is of following order:

$$\|L^{-1}D\|_{ref,L^2_\xi} \le O(1)\,\epsilon^2\, e^{-\mathscr{C}_3\epsilon|x|}.$$

Then, by the Schwartz inequality $C>1$ exists such that for any $\gamma_1\in(0,1)$,

$$\mathscr{I}_8 \le C\int_0^\tau\int_{-\infty}^{\infty}\left(\gamma_1\ \langle W_0|W_0\rangle\,\epsilon\,|\partial_x\lambda_3| + \frac{\epsilon}{\gamma_1}\Big(\langle J_1|J_1\rangle + \langle \boldsymbol{h}|\boldsymbol{h}\rangle\Big)\right) dxdt.$$

Finally, from the above estimates on $\mathscr{I}_i$'s we conclude that there exists $C>0$ such that for any $\gamma\in(0,1)$,

$$\begin{aligned}&\frac{1}{2}\int_{-\infty}^{\infty}\langle W_0|W_0\rangle + \Big\langle(\xi^1-s)W_0|L^{-1}J_1\Big\rangle dx\Big|_{t=0}^{t=\tau}\\ &\quad +\int_0^\tau\int_{-\infty}^{\infty}|\partial_x\lambda_3|\{(w_3)^2 - C(w_1^2+w_2^2)\}dxdt\\ &\quad -\int_0^\tau\int_{-\infty}^{\infty}\Big\langle \boldsymbol{P}_1\xi^1\boldsymbol{P}_0\partial_x W_0|L^{-1}|\boldsymbol{P}_1\xi^1\boldsymbol{P}_0\partial_x W_0\Big\rangle dxdt\\ &\le \int_0^\tau\int_{-\infty}^{\infty} C\gamma|\partial_x\lambda_3|\ \|W_0\|^2_{L^2_\xi} + \frac{C}{\gamma}\|J_1\|^2_{ref,L^2_\xi}dxdt.\end{aligned}\tag{9.3d}$$

Transversal Estimates. In (9.3d), the non-positive double integral $\int_0^\tau\int_{-\infty}^{\infty}|\partial_x\lambda_3|$ $(-Cw_1^2 - Cw_2^2)dxdt$ remains to be estimated. For this, we consider the following double integration:

$$\mathscr{G} \equiv \int_0^\tau\int_{-\infty}^{\infty}\partial_x\lambda_3\ \Big\langle W_0|(\xi^1-s)W_0\Big\rangle dxdt.$$

Since the shock wave is a 3-shock wave (8.2), there exists $C>0$ such that

$$\begin{aligned}\mathscr{G} &= \int_0^\tau\int_{-\infty}^{\infty}\partial_x\lambda_3\sum_{j=1}^{3}\lambda_j(w_j)^2dxdt\\ &\ge \int_0^\tau\int_{-\infty}^{\infty}|\partial_x\lambda_3|\Big(\|\lambda_2\|_\infty((w_1)^2+(w_2)^2) - C\epsilon(w_3)^2\Big)\ dxdt.\end{aligned}$$

Remark 9.2. The consideration of a transversal wave estimate originates from the energy estimate for the stability analysis of a viscous shock profile, [6]. Here, we have rewritten it in a compact format in order to be able to apply it to the Boltzmann equation.

Use integration by parts to yield

$$\begin{aligned}
\mathscr{G} &= -\int_0^\tau \int_{-\infty}^\infty \lambda_3 \partial_x \left\langle W_0 | (\xi^1 - s) W_0 \right\rangle dx \qquad (9.4)\\
&= -\int_0^\tau \int_{-\infty}^\infty \lambda_3 2 \left\langle W_0 | (\xi^1 - s) \boldsymbol{P}_0 \partial_x W_0 \right\rangle + O(1)\epsilon^3 e^{-\mathscr{C}_1 \epsilon |x|} \ \langle W_0 | W_0 \rangle \, dxdt \\
&= -\int_0^\tau \int_{-\infty}^\infty \lambda_3 2 \left\langle W_0 | - \partial_t W_0 - (\xi^1 - s) J_1 \right\rangle + O(1)\epsilon^3 e^{-\mathscr{C}_1 \epsilon |x|} \ \langle W_0 | W_0 \rangle \, dxdt \\
&= \int_{-\infty}^\infty \lambda_3 \langle W_0 | W_0 \rangle \, dx \Big|_{t=0}^{t=\tau} \\
&\quad + \int_0^\tau \int_{-\infty}^\infty 2\lambda_3 \left\langle W_0 | (\xi^1 - s) J_1 \right\rangle + O(1)\epsilon^3 \ e^{-\mathscr{C}_1 \epsilon |x|} \ \langle W_0 | W_0 \rangle \, dxdt,
\end{aligned}$$

and so

$$\begin{aligned}
&\int_{-\infty}^\infty -\lambda_3 \langle W_0 | W_0 \rangle \, dx \Big|_{t=0}^{t=\tau} + \int_0^\tau \int_{-\infty}^\infty -\partial_x \lambda_3 \ ((w_1)^2 + (w_2)^2 - C\epsilon (w_3)^2) dxdt \\
&\qquad \le \int_0^\tau \int_{-\infty}^\infty 2\lambda_3 \langle W_0 | J_1 \rangle + O(1) \ \epsilon^3 e^{-\mathscr{C}_1 \epsilon |x|} \langle W_0 | W_0 \rangle \, dxdt.
\end{aligned}$$

In the above estimates, the term $\int_0^\tau \int_{-\infty}^\infty 2\lambda_3 \langle W_0 | J_1 \rangle \, dxdt$ can be bounded by $O(1) \ \epsilon \ (\mathscr{I}_3 + \cdots + \mathscr{I}_8)$. It follows that

$$\begin{aligned}
&\int_0^\tau \int_{-\infty}^\infty - \partial_x \lambda_3 \ ((w_1)^2 + (w_2)^2 - C\epsilon (w_3)^2) dxdt \\
&\qquad \le O(1)\epsilon \int_{-\infty}^\infty \left(\| W_0 \|^2_{L^2_\xi} + \| J_1 \|^2_{ref, L^2_\xi} \right) \Big|_{t=0} \\
&\qquad\quad + \left(\| W_0 \|^2_{L^2_\xi} + \| J_1 \|^2_{ref, L^2_\xi} \right) \Big|_{t=\tau} dx \\
&\qquad\quad + O(1)\epsilon^3 \gamma \int_0^\tau \int_{-\infty}^\infty e^{-\mathscr{C}_1 \epsilon |x|} \| W_0 \|^2_{L^2_\xi} dxdt \\
&\qquad\quad + O(1) \frac{\epsilon}{\gamma} \int_0^\tau \int_{-\infty}^\infty \| J_1 \|^2_{ref, L^2_\xi} dxdt. \qquad (9.5)
\end{aligned}$$

Take γ to satisfy

$$\epsilon \ll \gamma \ll 1;$$

and consider the combination

$$(9.2) + \frac{1}{\gamma}(9.5) + \int_0^\tau \int_{-\infty}^\infty \langle J, (7.2) \rangle_{ref} \, dxdt.$$

(Here, the estimate for $\int_0^\tau \int_{-\infty}^\infty \langle J, (7.2)\rangle_{ref}\, dxdt$ is almost identical to (5.17) and is omitted.) This results in the first order energy estimate

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left[\frac{1}{2} + O(1)\epsilon\right] \|W_0\|^2_{L^2_\xi} + \frac{1}{\gamma C}(1 + O(1)\epsilon)\|J\|^2_{ref,L^2_\xi} dx\Big|_{t=\tau} \\
&\quad - \int_0^\tau \int_{-\infty}^\infty \Big\langle \boldsymbol{P}_1(\xi^1 - s)J_0|L^{-1}|\boldsymbol{P}_1(\xi^1 - s)J_0\Big\rangle dxdt \\
&\quad + \int_0^\tau \int_{-\infty}^\infty \frac{|\partial_x \lambda_3|}{2}\left((w_3)^2 + \frac{\|\lambda_2\|_\infty}{2\gamma}((w_1)^2 + (w_2)^2)\right)\, dxdt \\
&\le 2\int_{-\infty}^{\infty} \left[\frac{1}{2} + O(1)\epsilon\right] \|W_0\|^2_{L^2_\xi} + \frac{1}{\gamma C}(1 + O(1)\epsilon)\|J\|^2_{ref,L^2_\xi} dx\Big|_{t=0}.
\end{aligned}$$

This corresponds to the energy estimate of (5.14) with corrections due to the presence of the shock layer.

With the shock correcting terms estimated in (9.5), we can apply the same analysis leading to (5.22) to conclude that, for given σ and γ sufficiently small,

$$\begin{aligned}
&\sum_{0\le i\le 6} \gamma^{-i} \int_0^\tau \int_{\boldsymbol{R}} \Big\langle \partial_x^i W_0|\partial_x^i (W_{0t} + \boldsymbol{P}_0(\xi^1 - s)W_{0x} + \boldsymbol{P}_0(\xi^1 - s)J_1\Big\rangle dxdt \\
&+ \sum_{0\le i\le 6} \gamma^{-i} \int_0^\tau \int_{\boldsymbol{R}} \langle \partial_x^i W_0|\partial_x^i \boldsymbol{P}_0(\xi^1 - s)\bar{L}^{-1}((J_{1t} + \boldsymbol{P}_1(\xi^1 - s)J_{1x} \\
&+ \boldsymbol{P}_1(\xi^1 - s)J_{0x} - \rho\bar{L}J_1 - \boldsymbol{N}(J)))\rangle dxdt \\
&+ \sum_{0\le i\le 6} \gamma^{-i-1} \int_0^\tau \int_{\boldsymbol{R}} \Big\langle \partial_x^i J, \partial_x^i (J_t + \xi J_x - LJ - \mathcal{N}(J))\Big\rangle_{ref} dxdt \\
&+ \sum_{0\le i\le 6} \gamma^{-i} \int_0^\tau \int_{\boldsymbol{R}} \Big\langle \partial_x^i \partial_t W_0|\partial_x^i \partial_t (W_{0t} + \boldsymbol{P}_0(\xi^1 - s)W_{0x} + \boldsymbol{P}_0(\xi^1 - s)J_1\Big\rangle dxdt \\
&+ \sum_{0\le i\le 6} \gamma^{-i} \int_0^\tau \int_{\boldsymbol{R}} \langle \partial_x^i \partial_t W_0|\partial_x^i \partial_t \boldsymbol{P}_0(\xi^1 - s)\bar{L}^{-1} \\
&\quad \times ((J_{1t} + \boldsymbol{P}_1(\xi^1 - s)J_{1x} + \boldsymbol{P}_1(\xi^1 - s)J_{0x} - \rho\bar{L}J_1 - \boldsymbol{N}(J)))\rangle dxdt \\
&+ \sum_{0\le i\le 6} \gamma^{-i-1} \int_0^\tau \int_{\boldsymbol{R}} \Big\langle \partial_x^i \partial_t J, \partial_x^i \partial_t (J_t + \xi J_x - LJ - \mathcal{N}(J))\Big\rangle_{ref} dxdt \\
&+ \frac{1}{\gamma}\int_0^\tau \int_{-\infty}^\infty \partial_x \lambda_3\, \Big\langle W_0|(\xi^1 - s)W_0\Big\rangle dxdt - 2\lambda_3 \Big\langle W_0|W_t + (\xi^1 - s)J_1\Big\rangle \\
&+ O(1)\,\epsilon^3 e^{-\mathscr{C}_1 \epsilon|x|}\,\langle W_0|W_0\rangle\, dxdt = 0.
\end{aligned}$$

Finally, this combination yields the higher order energy estimate:
There exists $C > 0$ such that

$$
\begin{aligned}
&\frac{1}{C}\sum_{0\le i\le 6}\int_{\boldsymbol{R}} \\
&\left(\gamma^{-i}\left(\|\partial_x^i W_0\|^2_{L^2_\xi}+\|\partial_x^i\partial_t W_0\|^2_{L^2_\xi}\right)+\gamma^{-i-1}\left(\|\partial_x^i J\|^2_{ref,L^2_\xi}+\|\partial_x^i\partial_t J\|^2_{ref,L^2_\xi}\right)\right)\Bigg|_{t=\tau} dx \qquad (9.6)\\
&\quad +\frac{\epsilon^2}{C}\int_0^\tau\int_{\boldsymbol{R}}\|W_0\|^2_{L^2_\xi}\, e^{-\mathscr{C}_1\,\epsilon|x|}\,dxdt\\
&\quad +\sum_{0\le i\le 6}\int_0^\tau\int_{\boldsymbol{R}}\gamma^{-i}\left(\|\partial_x^i \boldsymbol{h}\|^2_{L^2_\xi}+\|\partial_x^i\partial_t \boldsymbol{h}\|^2_{L^2_\xi}\right)dxdt\\
&\quad +\sum_{0\le i\le 6}\int_0^\tau\gamma^{-i-1}\left(\|(1+|\xi|)^{\frac12}\partial_x^i J_1\|^2_{ref,L^2_\xi}+\|(1+|\xi|)^{\frac12}\partial_x^i\partial_t J_1\|^2_{ref,L^2_\xi}\right)dxdt\\
&\le C\sum_{0\le i\le 6}\int_{\boldsymbol{R}}\left(\gamma^{-i}\left(\|\partial_x^i W_0\|^2_{L^2_\xi}+\|\partial_x^i\partial_t W_0\|^2_{L^2_\xi}\right)+\gamma^{-i-1}\left(\|\partial_x^i J\|^2_{ref,L^2_\xi}\right.\right.\\
&\quad \left.\left.+\|\partial_x^i\partial_t J\|^2_{ref,L^2_\xi}\right)\right)\Bigg|_{t=0} dx.
\end{aligned}
$$

With this, (5.23), and the Sobolev inequality, the smallness assumption (9.1) is justified when ς_0 is sufficiently small. Thus, nonlinear stability of the shock profile follows.

10. Positivity of Shock Profiles

We consider the initial value problem

$$
\begin{cases} F_t+(\xi^1-s)F_x=Q(F,F),\\ F(x,0)\equiv\phi_{\mathscr{T}}(x)>0.\end{cases}
$$

From Lemma 7.3, $J\equiv F-\phi$ satisfies

$$
\int_{-\infty}^{\infty}\int_{\boldsymbol{R}^3}\chi_i J(x,0,\xi)\,d^3\xi dx=0,\ \text{for } i=0,\cdots,4.
$$

The stability theory yields that

$$
\lim_{t\to\infty}J(x,t,\xi)=0 \text{ or equivalently } \lim_{t\to\infty}F(x,t,\xi)=\phi(x,\xi). \qquad (10.1)
$$

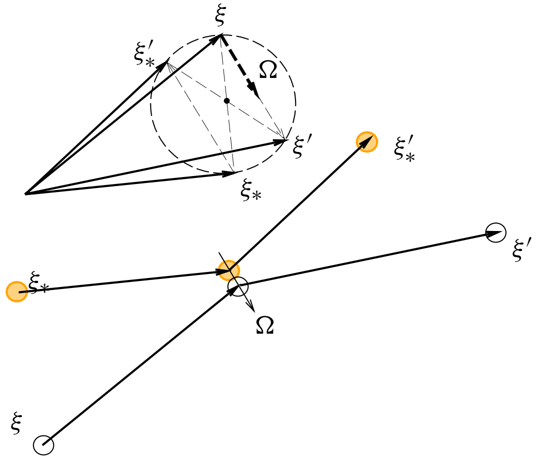
We break the collision operator into lost-gain parts:

$$
F_t+(\xi^1-s)F_x+Q_-(F,F)=Q_+(F,F), \qquad (10.2a)
$$

$$
\begin{aligned}
Q_-(F,F)&\equiv\int_{\boldsymbol{R}^3\times S^2}F(x,t,\xi_*)\,C(\Omega,\xi-\xi_*)\,d\xi_*d\Omega\,F(x,t,\xi),\\
&\equiv\Gamma(F)\cdot F,\\
Q_+(F,F)&\equiv\int_{\boldsymbol{R}^3\times S^2}F(x,t,\xi')F(x,t,\xi'_*)\,C(\Omega,\xi-\xi_*)\,d\xi_*d\Omega, \qquad (10.2b)
\end{aligned}
$$

where

$$\begin{cases} \xi' = \xi + (\Omega \cdot (\xi_* - \xi))\, \Omega, \\ \xi'_* = \xi_* - (\Omega \cdot (\xi_* - \xi))\, \Omega, \\ \Omega \in \boldsymbol{S}^2. \end{cases}$$



and $C(\Omega, \xi - \xi_*)$ is the angular cut-off function by [7]. In our case, the function $C(\Omega, \xi - \xi_*)$ for a hard sphere collision is

$$C(\Omega, \xi - \xi_*) \equiv |\Omega \cdot (\xi - \xi_*)|. \tag{10.3}$$

One can rewrite (10.2a) as follows:

$$F_t + (\xi^1 - s) F_x + \Gamma(F)\ F = Q_+(F, F). \tag{10.4}$$

This yields the representation of the solution:

$$\begin{aligned} F(x, t, \xi) = \int_0^t & e^{-\int_\tau^t \Gamma(F)(x - (\xi^1 - s)(t - \tau'), \tau', \xi)\, d\tau'} \\ & \times Q_+(F, F)(x - (\xi^1 - s)(t - \tau), \tau, \xi)\, d\tau \\ & + e^{-\int_0^t \Gamma(F)(x - (\xi^1 - s)(t - \tau'), \tau', \xi) d\tau'} F(x - (\xi^1 - s)t, 0, \xi). \end{aligned} \tag{10.5}$$

Here, the functions $\Gamma(F)$ and $Q_+(F, F)$ are positive when the function F is positive and integrable in ξ. Since the initial data of F is positive, the global existence of F and (10.5) yield that

$$F > 0.$$

This and (10.1) result in

$$\phi(x, \xi) = \lim_{t \to \infty} F(x, \xi, t) \geq 0.$$

From the stability analysis, ϕ is a small perturbation of a local Maxwellian profile. Thus, we conclude that there exists a sufficiently large $R > 0$ such that

$$\phi(x, \xi) > 0 \text{ for all } |\xi| < R. \tag{10.6}$$

We now show that ϕ is strictly positive for all $\xi \in \boldsymbol{R}^3$. This is proved by contradiction. Suppose that

$$\phi(x_0, \xi_0) = 0 \text{ for some } (x_0, \xi_0) \in \boldsymbol{R} \times \boldsymbol{R}^3. \tag{10.7}$$

Since $\phi(x, \xi)$ is a stationary solution of the evolution equation,

$$\phi_t(x, \xi) + (\xi^1 - s)\phi_x + Q_-(\phi, \phi) = Q_+(\phi, \phi).$$

We have the integral representation of $\phi(x, \xi)$:

$$\begin{aligned}\phi(x, \xi) = \int_0^t & e^{-\int_\tau^t \Gamma(\phi)(x-(\xi^1-s)(t-\tau'),\xi)\, d\tau'} Q_+(\phi, \phi)(x - (\xi^1 - s)(t - \tau), \xi)\, d\tau \\ & + e^{-\int_0^t \Gamma(\phi)(x-(\xi^1-s)(t-\tau'),\xi) d\tau'} \phi(x - (\xi^1 - s)t, \xi).\end{aligned}$$

Since $\phi \geq 0$, the above double integrals are non-negative. With the assumption (10.7), both of the above double integrals are zero for $(x, \xi) = (x_0, \xi_0)$.

Thus

$$\begin{cases}\phi(x_0 - (\xi_0^1 - s)t, \xi_0) \equiv 0, \\ Q_+(\phi, \phi)|_{(x,\xi)=(x_0-(\xi_0^1-s)t,\xi_0)} & = 0.\end{cases}$$

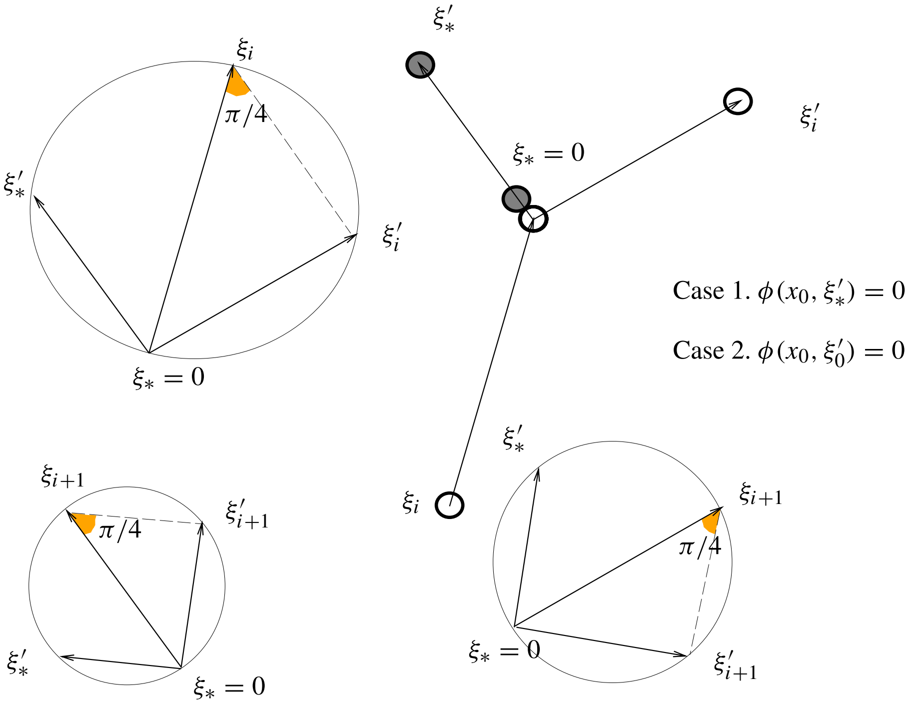
Set $t \equiv 0$ to yield

$$0 = \int_{\boldsymbol{R}^3 \times S^2} \phi(x_0, \xi_0')\phi(x_0, \xi_*')C(\Omega, \xi_0 - \xi_*) d\xi_* d\Omega.$$

Here, the function $C(\Omega, \xi - \xi_*)$ defined in the operator Q_+ in (10.2b) is a positive function, except for the case when $\xi_0 - \xi_*$ and $\xi_0' - \xi_0$ are orthogonal,

$$\phi(x_0, \xi_0')\phi(x_0, \xi_*') = 0 \text{ for } (\xi_0 - \xi_*) \cdot (\xi_0' - \xi_0) \neq 0. \tag{10.8}$$

We now set up a reduction procedure to find a sequence $\xi_0, \xi_1, \cdots$ with $\lim_{i \to \infty} \xi_i = 0$ and $\phi(\xi_0, \xi_i) = 0$. This would lead to contradiction to (10.6).



Case 1 Case 2

Suppose that (x_0, ξ_i) is given and $\phi(x_0, \xi_i) = 0$. Consider the collision of a particle with velocity ξ_i to a stationary particle with velocity $\xi_* = 0$ with $\pi/4$ scattering angle, the angle between $\xi_i - \xi_*$ and ξ'_i:

$$(\xi'_i - \xi_i) \cdot (\xi_i - \xi_*) = \cos\frac{\pi}{4} \|\xi'_i - \xi_i\| \cdot \|\xi_i - \xi_*\| \neq 0.$$

Apply (10.8) to either $\phi(\xi'_i, x_0) = 0$ or $\phi(\xi'_*, x_0) = 0$. Take ξ_{i+1} to be ξ'_i or ξ'_*, $\phi(\xi_{i+1}, x_0) = 0$. By the choice of the angle, we have $|\xi_{i+1}| = |\xi_i|/\sqrt{2} = 2^{-(i+1)/2}|\xi_0|$. This leads to a contradiction to (10.6) and we have thus proved the positivity of the shock profile

$$\phi(x, \xi) > 0 \text{ for all } (x, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^3.$$

Appendix A. Chapman-Enskog Expansion

Consider the Boltzmann equation

$$\partial_t F + \xi \cdot \nabla_x F = \frac{1}{\kappa}\, Q(F, F), \tag{A.1}$$

where $\kappa > 0$ is the mean free path.

Expand the solution F and the operator ∂_t:

$$F = \mathsf{F}_0 + \kappa \mathsf{F}_1 + \kappa^2 \mathsf{F}_2 + \dots,$$
$$\frac{\partial}{\partial t} = \frac{\partial_0}{\partial t} + \kappa \frac{\partial_1}{\partial t} + \kappa^2 \frac{\partial_2}{\partial t} + \dots,$$

where $\frac{\partial_i}{\partial t}$ are differential operators of x on the fluid variable $\rho, u^i, T, \dots$. The leading term F_0:

$$\mathsf{F}_0(\xi, x, t) \equiv \rho \, \frac{e^{-\frac{|\xi - u|^2}{2RT}}}{\sqrt{(2\pi RT)^3}}$$

$$\begin{cases} \dfrac{\partial_0}{\partial t}\rho \equiv -div\, m, \\ \dfrac{\partial_0}{\partial t} m^i \equiv -\displaystyle\sum_{j=1}^{3} \frac{\partial}{\partial x^j} m^i u^j - \frac{\partial}{\partial x^i} p, \\ \dfrac{\partial_0}{\partial t}\, E \equiv -\displaystyle\sum_{j=1}^{3} \frac{\partial}{\partial x^j}\, [E + p]\, u^j, \end{cases}$$

$$m^i \equiv \rho u^i, \; E \equiv \rho \left(\frac{|u|^2}{2} + \mathscr{E} \right), \; \mathscr{E} \equiv \frac{3}{2} RT, \; p \equiv \rho RT. \; (R \equiv 1).$$

F_1 is uniquely solved by:

$$Q(\mathsf{F}_0, \mathsf{F}_1) + Q(\mathsf{F}_1, \mathsf{F}_0) = \frac{\partial_0}{\partial t} \mathsf{F}_0 + \xi \cdot \nabla_x \mathsf{F}_0, \tag{A.2}$$
$$\int_{\boldsymbol{R}^3} \mathsf{F}_1 d\xi = 0,$$
$$\int_{\boldsymbol{R}^3} \xi^i \mathsf{F}_1 d\xi = 0 \text{ for } i = 1, 2, 3,$$
$$\int_{\boldsymbol{R}^3} |\xi|^2 \mathsf{F}_1 d\xi = 0.$$

Notice that

$$\mathsf{F}_1 \textit{ is purely microscopic; and } |\partial^k_{x^i} \mathsf{F}_1| \le O(1) \sup_{1 \le j \le k} |\partial^k_{x^i} \partial_{x^j} \mathsf{F}_0|. \tag{A.3}$$

The operator $\frac{\partial_1}{\partial t}$ is defined by

$$\frac{\partial_1}{\partial t} \rho \equiv - \int_{\boldsymbol{R}^3} \xi \cdot \nabla_x \mathsf{F}_1 d\xi = 0,$$
$$\frac{\partial_1}{\partial t}\, u^i \rho \equiv - \int_{\boldsymbol{R}^3} \xi^i \xi \cdot \nabla_x \mathsf{F}_1 d\xi$$
$$= \sum_{j=1}^{3} \frac{\partial}{\partial x^j} \left(\mu(T) \left[\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} - \frac{2}{3} \sum_{k=1}^{3} \frac{\partial u^k}{\partial x^k}\, \delta^i_j \right] \right) \text{ for } i = 1, 2, 3,$$

$$\frac{\partial_1}{\partial t}\,\rho\left(\frac{|u|^2}{2}+\mathscr{E}\right)\equiv-\int_{\boldsymbol{R}^3}|\xi|^2\,\xi\cdot\nabla_x\mathsf{F}_1 d\xi$$
$$=\sum_{1\le j,k\le 3}\frac{\partial}{\partial x^j}\,\mu(T)\left[u^k\left(\frac{\partial u^k}{\partial x^j}+\frac{\partial u^j}{\partial x^k}\right)-\frac{2}{3}u^j\frac{\partial u^k}{\partial x^j}\right]$$
$$+\sum_{j=1}^{3}\frac{\partial}{\partial x^j}\lambda(T)\frac{\partial T}{\partial x^j},$$

where $\mu(T) > 0$ is the viscosity coefficient and $\lambda(T) > 0$ is the heat conductivity coefficient. These dissipation coefficients can be estimated for the hard sphere, (B.3). From this definition of $\frac{\partial_1}{\partial t}$,

$$\frac{\partial_1}{\partial t}\mathsf{F}_0+\xi\cdot\nabla_x\mathsf{F}_1 \textit{ is purely microscopic.} \tag{A.4}$$

From (A.2), (A.3), and (A.4),

$$\left(\left[\frac{\partial_0}{\partial t}+\frac{\kappa\partial_1}{\partial t}\right]\mathsf{F}_0+\xi\cdot\nabla_x(\mathsf{F}_0+\kappa\mathsf{F}_1)\right)$$
$$-\frac{1}{\kappa}\left(Q(\mathsf{F}_0+\kappa\mathsf{F}_1,\mathsf{F}_0+\kappa\mathsf{F}_1)-Q(\kappa\mathsf{F}_1,\kappa\mathsf{F}_1)\right)\equiv\mathscr{S}, \tag{A.5}$$
$$\mathscr{S}:Purelymicroscopic. \tag{A.6}$$

The first two terms in the expansion of $\frac{\partial}{\partial t}$

$$\begin{cases}\left(\frac{\partial}{\partial t}-\frac{\partial_0}{\partial t}-\kappa\,\frac{\partial_1}{\partial t}\right)\rho=0,\\ \left(\frac{\partial}{\partial t}-\frac{\partial_0}{\partial t}-\kappa\,\frac{\partial_1}{\partial t}\right)\rho u^i=0\\ \left(\frac{\partial}{\partial t}-\frac{\partial_0}{\partial t}-\kappa\,\frac{\partial_1}{\partial t}\right)\rho\left(\frac{|u|^2}{2}+\mathscr{E}\right)=0,\end{cases}$$

give the compressible Navier-Stokes equation:

$$\begin{cases}\rho_t+div\ m=0,\\ m_t^i+\sum_{j=1}^{3}(m^iu^j)_{x^j}+p_{x^i}=\kappa\sum_{j=1}^{3}\left(\mu(T)\left[u^i_{x^j}+u^j_{x^i}-\frac{2}{3}\sum_{k=1}^{3}u^k_{x^k}\,\delta^i_j\right]\right)_{x^j},\quad i=1,2,3,\\ E_t+\sum_{j=1}^{3}\left(u^j\,[E+p]\right)_{x^j}=\kappa\sum_{1\le j,k\le 3}\left(\mu(T)\left[u^k\left(u^k_{x^j}+u^j_{x^k}\right)-\frac{2}{3}u^ju^k_{x^j}\right]\right)_{x^j}\\ \qquad+\kappa\sum_{j=1}^{3}\left(\lambda(T)\ T_{x^j}\right)_{x^j}.\end{cases} \tag{A.7}$$

Let (ρ, m, E) be the solution of (A.7). From this,

$$\frac{\partial}{\partial t}\mathsf{F}_0-\left(\frac{\partial_0}{\partial t}+\kappa\frac{\partial_1}{\partial t}\right)\mathsf{F}_0=0.$$

From this and (A.5),

$$\frac{\partial}{\partial t}(\mathsf{F}_0 + \kappa \mathsf{F}_1) + \xi \cdot \nabla_x (\mathsf{F}_0 + \kappa \mathsf{F}_1) - \frac{1}{\kappa} Q(\mathsf{F}_0 + \kappa \mathsf{F}_1, \mathsf{F}_0 + \kappa \mathsf{F}_1) = \mathsf{S}, \quad \text{(A.8)}$$
$$\mathsf{S} \equiv \kappa \frac{\partial}{\partial t}\mathsf{F}_1 + \kappa Q(\mathsf{F}_1, \mathsf{F}_1) + \mathscr{S},$$
$$\mathsf{S} : \textit{ Purely microscopic },$$
$$|\mathsf{S}| \leq O(1)\, \epsilon^3\, e^{-\frac{9}{10}(\mu + \frac{\lambda}{5})|u_+ - s + \sqrt{5T_+/3}|\, |x|}.$$

Let $(\rho, m, E)(x,t) = (\bar\rho, \bar m, \bar E)(x^1 - st)$ be a travelling wave solution of (A.7) connecting the two end states $(\rho_\pm, m_\pm, E_\pm)$ of a planar entropy shock wave in the x^1-direction; and let the length of the mean free path κ:

$$\kappa \equiv 1;$$

and the equation of the shock profile is

$$\begin{cases} -s\bar\rho_{x^1} + \bar m^1_{x^1} = 0, \\ -s\bar m^1_{x^1} + (\bar m^1 \bar u^1)_{x^1} + \bar p_{x^1} = \frac{4}{3}\left(\mu(\bar T)\bar u^1_{x^1}\right)_{x^1}, \\ -s\bar E_{x^1} + \left(\bar u^1\left[\bar E + \bar p\right]\right)_{x^1} = \frac{4}{3}\left(\mu(\bar T)\bar u^1 \bar u^1_{x^1}\right)_{x^1} + \left(\lambda(\bar T)\, \bar T_{x^1}\right)_{x^1}, \end{cases} \quad \text{(A.9)}$$
$$\lim_{x\to\pm\infty}(\bar\rho, \bar m, \bar E)(x) = (\rho_\pm, m_\pm, E_\pm).$$

We will view $\mathsf{A} = \mathsf{F}_0 + \mathsf{F}_1$ as the approximate travelling wave for Boltzmann equation. The truncation error is then

$$\mathsf{S} \equiv (\xi^1 - s)(\mathsf{F}_0 + \mathsf{F}_1)_{x^1} - Q(\mathsf{F}_0 + \mathsf{F}_1, \mathsf{F}_0 + \mathsf{F}_1), \quad \text{(A.10)}$$
$$\mathsf{S} : \textit{ Purely Microscopic}, \quad \text{(A.11)}$$
$$|\mathscr{S}| \leq O(1)\, \epsilon^3\, e^{-\frac{9}{10}(\mu + \frac{\lambda}{5})|u_+ - s + \sqrt{5T_+/3}|\, |x|}, \quad \text{(A.12)}$$
$$\mathsf{A} \equiv \mathsf{F}_0 + \mathsf{F}_1. \quad \text{(A.13)}$$

When the strength $|\rho_- - \rho_+| + |m_- - m_+| + |E_- - E_+|$ of the shock wave is sufficiently small, the acoustic speed of the travelling solution is monotone: (Here, the shock is a 3-shock.)

$$\partial_x\left(\bar u^1 + \sqrt{\frac{5\bar T}{3}}\right) < 0. \quad \text{(A.14)}$$

Under the same setting as in (6.3), there exist C_u, c_l, and c_u satisfying

for $x^1 \geq 0$:

$$\begin{cases} \epsilon^2 \mathsf{c}_l e^{-\frac{10}{9}\mathsf{e}\epsilon x^1} \leq -\partial_x\left(\bar u^1 + \sqrt{\frac{5\bar T}{3}}\right) \leq \epsilon^2 \mathsf{c}_u e^{-\frac{9}{10}\mathsf{e}\epsilon x^1}, \\ |\mathsf{S}| \leq \epsilon^3\, \mathsf{C}_U e^{-\frac{9}{10}\mathsf{e}\epsilon x^1}, \end{cases} \quad \text{(A.15a)}$$

for $x^1 \leq 0$:

$$\begin{cases} \epsilon^2 \mathsf{c}_l e^{\frac{10}{9}\mathsf{e}\epsilon x^1} \leq -\partial_x\left(\bar u^1 + \sqrt{\frac{5\bar T}{3}}\right) \leq \epsilon^2 \mathsf{c}_u e^{\frac{9}{10}\mathsf{e}\epsilon x^1}, \\ |\mathsf{S}| \leq \epsilon^3\, \mathsf{C}_U e^{\frac{9}{10}\mathsf{e}\epsilon x^1}, \end{cases} \quad \text{(A.15b)}$$

where

$$\mathsf{e} \equiv \lim_{\epsilon\to 0+} \frac{3}{4}\left(\mu + \frac{\lambda}{5}\right)\frac{|u_+ - s + \sqrt{5T_+/3}|}{\epsilon} \sim \frac{1}{\sqrt{\pi}}\frac{105}{256}\lim_{\epsilon\to 0+}\frac{|u_+ - s + \sqrt{5T_+/3}|}{\epsilon}. \tag{A.15c}$$

The last estimate is for the hard sphere, see (B.3) of the next section.

These facts about travelling wave solutions of the compressible Navier-Stokes equation can be easily verified.

Appendix B. Estimates on Collision Operators

From now on we will consider only the hard spheres. The collision operator can be written as follows see [8, 7]:

$$Lh(\xi) \equiv -\nu(\xi)h(\xi) + \int_{\boldsymbol{R}^3}(-k_1(\xi,\eta) + k_2(\xi,\eta))h(\eta)\,d\eta, \tag{B.1}$$

$$Lh \equiv -\nu h - K_1 h + K_2 h. \tag{B.2}$$

$$\begin{cases} \nu(\xi) \equiv \sqrt{2\pi}b_0\left(e^{-\frac{|\xi|^2}{2}} + |\xi|^2\int_0^1 e^{-\frac{u^2|\xi|^2}{2}}\,du\right), \\ k_1(\xi,\eta) = \dfrac{b_0}{2\sqrt{2\pi}}|\xi-\eta|e^{-\frac{|\xi|^2+|\eta|^2}{4}}, \\ k_2(\xi,\eta) = \dfrac{2b_0}{\sqrt{2\pi}}\dfrac{1}{|\xi-\eta|}e^{-\frac{|\xi-\eta|^2}{8} - \frac{1}{8}\frac{(|\xi|^2-|\eta|^2)^2}{|\xi-\eta|^2}}, \end{cases}$$

here b_0 is a positive constant. (We set $b_0 = 1$.)

Under the normalized condition (6.3), the coefficient of viscosity, heat conductivity and minimum of collision frequency are, ([5, 7]):

$$\begin{cases} \mu \sim \dfrac{5}{16\sqrt{\pi}}, \\ \lambda \sim \dfrac{75}{64\sqrt{\pi}}, \\ \nu_0 \sim \sqrt{2\pi}. \end{cases} \tag{B.3}$$

From (B.3) and (A.15c), we obtain the crucial estimate on the strength of the collision frequency needed for the energy estimate in the next section:

$$\frac{8\mathsf{e}\epsilon}{15|\lambda_3^{\pm}|} < \frac{\nu_0}{12}. \tag{B.4}$$

Lemma B.1. *Let the function $C(\Omega, \xi - \xi_*)$ be the collision operator $Q(F,G)$ for the hard sphere model, (see (10.3)). There exists $\mathsf{K} > 0$ such that the following inequality holds for any g and h satisfying $\|(1+|\xi|)^{\frac{1}{2}}g\|_{L^2(\boldsymbol{R}^3)}$, $\|(1+|\xi|)^{\frac{1}{2}}h\|_{L^2(\boldsymbol{R}^3)} < \infty$:*

$$\begin{aligned}\int_{\boldsymbol{R}^3}\frac{Q(\omega_0 g,\omega_0 h)^2 + Q(\omega_0 h,\omega_0 g)^2}{(|\xi|+1)\omega_0(\xi)^2}d\xi &\le \mathsf{K}\int_{\boldsymbol{R}^3}(|\xi|+1)g(\xi)^2\,d\xi\cdot\int_{\boldsymbol{R}^3}h(\xi)^2\,d\xi \\ &\quad + \mathsf{K}\int_{\boldsymbol{R}^3}g(\xi)^2\,d\xi\cdot\int_{\boldsymbol{R}^3}(|\xi|+1)h(\xi)^2\,d\xi,\end{aligned} \tag{B.5}$$

where $\omega_0(\xi)$ is a given Maxwellian distribution.

Proof. Since (ξ', ξ'_*) given in (10.2b) is constructed to conserve the kinetic energy and momentum after collision,

$$|\xi|^2 + |\xi_*|^2 = |\xi'|^2 + |\xi'_*|^2.$$

From this,

$$\omega_0(\xi)\omega_0(\xi_*) = \omega_0(\xi')\omega_0(\xi'_*).$$

This results in

$$\begin{aligned}&\int_{\boldsymbol{R}^3}\int_{\mathsf{S}^2} \frac{g(\xi')h(\xi'_*)\omega_0(\xi')\omega_0(\xi'_*)C(\Omega, \xi - \xi_*)}{\omega_0(\xi)\sqrt{1+|\xi|}} d\Omega d\xi_* \\ &\quad = \int_{\boldsymbol{R}^3}\int_{\mathsf{S}^2} \frac{g(\xi')h(\xi'_*)\omega_0(\xi_*)C(\xi - \xi*, \Omega)}{\sqrt{1+|\xi|}} d\Omega d\xi_*.\end{aligned}$$

We also have

$$\begin{aligned}&\frac{1}{\omega_0(\xi)\sqrt{1+|\xi|}} Q(\omega_0 g, \omega_0 h) \\ &\quad = \frac{1}{\sqrt{1+|\xi|}} \int_{\boldsymbol{R}^3}\int_{\mathsf{S}^2} -g(\xi)h(\xi_*)\omega_0(\xi_*)C(\Omega, \xi - \xi_*) d\Omega d\xi_* \\ &\quad\quad + \frac{1}{\sqrt{1+|\xi|}} \int_{\boldsymbol{R}^3}\int_{\mathsf{S}^2} g(\xi')h(\xi'_*)\omega_0(\xi_*)C(\Omega, \xi - \xi_*) d\Omega d\xi_*. \qquad \text{(B.6)}\end{aligned}$$

By Hölder's inequality and by

$$\int\limits_{\boldsymbol{R}^3} \omega_0(\xi_*)C(\Omega, \xi - \xi_*)^2 \, d\xi_* = O(1)(1+|\xi|)^2,$$

it follows

$$\begin{aligned}&\int\limits_{\boldsymbol{R}^3} \frac{1}{|\xi|+1} \left(\int\limits_{\boldsymbol{R}^3 \times \boldsymbol{S}^2} g(\xi)h(\xi_*)\omega_0(\xi_*)C(\xi - \xi_*, \Omega) d\xi_* d\Omega \right)^2 d\xi \\ &\quad \le O(1) \int\limits_{\boldsymbol{R}^3} \frac{1}{|\xi|+1} \left(\int\limits_{\boldsymbol{R}^3 \times \boldsymbol{S}^2} g(\xi)^2 h(\xi_*)^2 \omega_0(\xi_*)C(\xi - \xi_*, \Omega)^2 d\xi_* d\Omega \right) d\xi \\ &\quad \le O(1) \int\limits_{\boldsymbol{R}^3 \times \boldsymbol{S}^2 \times \boldsymbol{R}^3} g(\xi)^2 h(\xi_*)^2 (1+|\xi|) d\xi d\Omega d\xi_* \\ &\quad = O(1) \|(1+|\xi|)^{\frac{1}{2}} g\|^2_{L^2(\boldsymbol{R}^3)} \, \|h\|^2_{L^2(\boldsymbol{R}^3)}. \qquad \text{(B.7)}\end{aligned}$$

By Hörder's inequality and by the following three properties of (ξ', ξ'_*):

$$\begin{cases} \text{The six dimensional volume element is invariant: } |d\xi d\xi_*| = |d\xi' d\xi'_*| \text{ with } \Omega \text{ fixed,} \\ C(\Omega, \xi - \xi_*) = C(\Omega, \xi' - \xi'_*), \\ \sup\limits_{\boldsymbol{R}^3 \times \boldsymbol{S}^2 \times \boldsymbol{R}^3} \dfrac{C(\xi - \xi_*, \Omega)^2 \omega_0(\xi_*)}{(1+|\xi'_*|)(1+|\xi|)} = O(1), \end{cases}$$

it follows

$$\begin{aligned}
&\int_{\boldsymbol{R}^3} \frac{1}{|\xi|+1} \left(\int_{\boldsymbol{R}^3\times\boldsymbol{S}^2} g(\xi')h(\xi'_*)\omega_0(\xi_*)C(\xi-\xi_*,\Omega)d\xi_* d\Omega \right)^2 d\xi \\
&\le O(1)\int_{\boldsymbol{R}^3} \frac{1}{|\xi|+1} \left(\int_{\boldsymbol{R}^3\times\boldsymbol{S}^2} g(\xi')^2 h(\xi'_*)^2\omega_0(\xi_*)C(\xi-\xi_*,\Omega)^2 d\xi_* d\Omega \right) d\xi \\
&= O(1) \int_{\boldsymbol{R}^3\times\boldsymbol{S}^2\times\boldsymbol{R}^3} \frac{g(\xi')^2 h(\xi'_*)^2\omega_0(\xi_*)C(\xi-\xi_*,\Omega)^2}{|\xi|+1} d\xi_* d\Omega d\xi \\
&= O(1) \int_{\boldsymbol{R}^3\times\boldsymbol{S}^2\times\boldsymbol{R}^3} \frac{g(\xi')^2 h(\xi'_*)^2\omega_0(\xi_*)C(\xi-\xi_*,\Omega)^2}{|\xi|+1} d\xi d\xi_* d\Omega \\
&= O(1) \int_{\boldsymbol{R}^3\times\boldsymbol{S}^2\times\boldsymbol{R}^3} g(\xi')^2(1+|\xi'|)h(\xi'_*)^2 d\xi d\xi_* d\Omega \\
&= O(1) \int_{\boldsymbol{R}^3\times\boldsymbol{S}^2\times\boldsymbol{R}^3} g(\xi')^2(1+|\xi'|)h(\xi'_*)^2 d\xi' d\xi'_* d\Omega \\
&= O(1)\|(1+|\xi|)^{\frac{1}{2}} g\|^2_{L^2(\boldsymbol{R}^3)} \; \|h\|^2_{L^2(\boldsymbol{R}^3)}.
\end{aligned} \tag{B.8}$$

We can conclude that

$$\int_{\boldsymbol{R}^3} \frac{Q(\omega_0 g,\omega_0 h)^2}{(1+|\xi|)\omega_0(\xi)^2} d\xi = O(1)\|(1+|\xi|)^{\frac{1}{2}} \; g\|^2_{L^2(\boldsymbol{R}^3)} \; \|h\|^2_{L^2(\boldsymbol{R}^3)}.$$

Similarly, we have

$$\int_{\boldsymbol{R}^3} \frac{Q(\omega_0 h,\omega_0 g)^2}{(1+|\xi|)\omega_0(\xi)^2} d\xi = O(1)\|(1+|\xi|)^{\frac{1}{2}} \; h\|^2_{L^2(\boldsymbol{R}^3)} \; \|g\|^2_{L^2(\boldsymbol{R}^3)}.$$

The above two estimates conclude this lemma. □

Lemma B.2. *For a given Maxwellian ω_0 and a given function d satisfying $d(\xi) \le \mathsf{D}_0\omega_0(\xi)^\alpha$ with $\alpha \in [1/6, 1/3]$, a linear operator $\mathbb{D}h \equiv \frac{1}{\omega_0}[Q(\omega_0 d,\omega_0 h)+Q(\omega_0 h,\omega_0 d)]$. There exists a constant K_0 satisfying*

$$\int_{\boldsymbol{R}^3} h\,\mathbb{D}h d\xi \le \mathsf{K}_0\,\mathsf{D}_0 \int_{\boldsymbol{R}^3} h(\xi)^2(1+|\xi|)\,d\xi.$$

Proof. By the Schwartz inequality and Lemma B.1, it follows

$$\begin{aligned}\int_{\boldsymbol{R}^3} h\,\mathbb{D}h\,d\xi &= \mathsf{D}_0\int_{\boldsymbol{R}^3}(1+|\xi|)^{\frac{1}{2}}h\,(1+|\xi|)^{-\frac{1}{2}}\frac{\mathbb{D}h}{\mathsf{D}_0}d\xi \qquad \text{(B.9)}\\ &\le \frac{\mathsf{D}_0}{2}\int_{\boldsymbol{R}^3}(1+|\xi|)h(\xi)^2+(1+|\xi|)^{-1}\left(\frac{\mathbb{D}h}{\mathsf{D}_0}\right)^2 d\xi\\ &= O(1)\mathsf{D}_0\int_{\boldsymbol{R}^3}(1+|\xi|)h(\xi)^2\,d\xi.\end{aligned}$$

This yields the lemma. □

Appendix C. Construction of Boltzmann Shock Profile

We now prove Theorem 8.1 on the accuracy of the Navier-Stokes shock profiles as an approximation of the Boltzmann shock profiles . We will use the weighted energy method. For other weighted energy methods for the study of boundary layers, see [1, 19] and references therein. We view the shock profile $\phi(x-st,\xi)$, $x\in\boldsymbol{R}^1$ of (6.1) as a perturbation of the Navier-Stokes profile A defined in (A.13):

$$\mathsf{V}\equiv\phi-\mathsf{A}.$$

The equation for V is

$$(\xi^1-s)\mathsf{V}_x-[Q(\mathsf{A},\mathsf{V})+Q(\mathsf{V},\mathsf{A})]=Q(\mathsf{V},\mathsf{V})-\mathsf{S}. \qquad \text{(C.1)}$$

Let $(\boldsymbol{P}_0^-,\boldsymbol{P}_1^-)$ and $(\boldsymbol{P}_0^+,\boldsymbol{P}_0^+)$. They are the macro-micro decompositions corresponding to the end Maxwellian states $\omega_-\equiv\omega(\xi;u_-,T_-)$ and $\omega_+\equiv\omega(\xi;u_+,T_+)$, respectively. We have therefore two sets of variables

$$\begin{cases}\mathsf{V}=\mathsf{V}_0^++\mathsf{V}_1^+,\\ \mathsf{V}=\mathsf{V}_0^-+\mathsf{V}_1^-,\\ \mathsf{V}_0^\pm\equiv\boldsymbol{P}_0^\pm\mathsf{V},\\ \mathsf{V}_1^\pm\equiv\boldsymbol{P}_1^\pm\mathsf{V};\end{cases}$$

and the operator $Q(\mathsf{V},\mathsf{A})+Q(\mathsf{A},\mathsf{V})$ can be written two different ways:

$$\begin{cases}Q(\mathsf{V},\mathsf{A})+Q(\mathsf{A},\mathsf{V})=L^+\mathsf{V}+\mathsf{R}^+\mathsf{V},\\ Q(\mathsf{V},\mathsf{A})+Q(\mathsf{A},\mathsf{V})=L^-\mathsf{V}+\mathsf{R}^-\mathsf{V},\\ L^+\mathsf{V}\equiv Q(\omega^+,\mathsf{V})+Q(\mathsf{V},\omega^+),\\ L^-\mathsf{V}\equiv Q(\omega^-,\mathsf{V})+Q(\mathsf{V},\omega^-),\\ \mathsf{R}^\pm\mathsf{V}\equiv Q(\mathsf{V},\mathsf{A}-\omega^\pm)+Q(\mathsf{A}-\omega^\pm,\mathsf{V}).\end{cases} \qquad \text{(C.2)}$$

Substitute the previous decomposition $\mathsf{A}=\mathsf{F}_0+\mathsf{F}_1$ into $\mathsf{R}^\pm\mathsf{V}$ to obtain, for a constant $\mathsf{C}>0$,

$$\begin{aligned}\mathsf{R}^\pm\mathsf{V}&=[Q(\mathsf{F}_0-\omega^\pm,\mathsf{V}_1^\pm)+Q(\mathsf{V}_1^\pm,\mathsf{F}_0-\omega^\pm)]+[Q(\mathsf{F}_1,\mathsf{V})+Q(\mathsf{V},\mathsf{F}_1)]\\ &\le \mathsf{C}\left(\epsilon|\mathsf{V}_1^\pm|+\epsilon^2|\mathsf{V}|\right). \qquad \text{(C.3)}\end{aligned}$$

The equations for $\mathsf{V}_0^\pm$ and V_1 are

$$\boldsymbol{P}_0^+(\xi^1 - s)\mathsf{V}_0^+ + \boldsymbol{P}_0^+(\xi^1 - s)\mathsf{V}_1^+ = 0, \tag{C.4a}$$
$$\boldsymbol{P}_1^+(\xi^1 - s)\mathsf{V}_{0x}^+ + \boldsymbol{P}_1^+(\xi^1 - s)\mathsf{V}_{1x}^+ - L^+\mathsf{V}_1^+ = \mathsf{R}^+\mathsf{V} + Q(\mathsf{V},\mathsf{V}) - \mathsf{S}, \tag{C.4b}$$

and

$$\boldsymbol{P}_0^-(\xi^1 - s)\mathsf{V}_0^- + \boldsymbol{P}_0^-(\xi^1 - s)\mathsf{V}_1^- = 0, \tag{C.5a}$$
$$\boldsymbol{P}_1^-(\xi^1 - s)\mathsf{V}_{0x}^- + \boldsymbol{P}_1^-(\xi^1 - s)\mathsf{V}_{1x}^- - L^-\mathsf{V}_1^- = \mathsf{R}^-\mathsf{V} + Q(\mathsf{V},\mathsf{V}) - \mathsf{S}. \tag{C.5b}$$

Equations (C.4a) and (C.5a) give algebraic relationships between $\mathsf{V}_0^\pm$ and $\mathsf{V}_1^\pm$. Similar to the matrix representations in (8.1) and the corresponding diagonalizations in (8.3) and (8.4), for $\mathsf{V}_0^\pm$ we have

$$\begin{cases} \mathsf{V}_0^\pm = \sum_{i=1}^3 \mathsf{v}_0^{i;\pm}\, \boldsymbol{r}_i^\pm,\ \mathsf{v}^{i;\pm} \in \boldsymbol{R}, \\ \boldsymbol{P}_0^\pm(\xi^1 - s)\, \boldsymbol{r}_i^\pm = \lambda_i^\pm\, \boldsymbol{r}_i^\pm,\ \lambda_i^\pm \in \boldsymbol{R}. \end{cases} \tag{C.6}$$

From the entropy condition in (8.2), there exists $\mathsf{c} > 0$ such that

$$\begin{cases} \lambda_i^+ < 0,\ \text{for } i = 1, 2, 3, \\ -\lambda_3^- < 0, \\ |\lambda_3^\pm| \ge \mathsf{c}\,\epsilon,\ \text{(because } \boldsymbol{P}_0(\xi^1 - s) \text{ is a non-singular matrix)}, \\ \mathsf{c}^{-1} > -\lambda_i^\pm \ge \mathsf{c} \text{ for } i = 1, 2. \end{cases} \tag{C.7}$$

From (C.6), (C.7), (C.4a) and (C.5a), there exists $\mathsf{C} > 0$ such that

$$\begin{cases} |\mathsf{v}_0^{i;\pm}| \le \mathsf{C}\sqrt{\langle \mathsf{V}_1^\pm | \mathsf{V}_1^\pm\rangle^\pm} \text{ for } i = 1, 2, \\ |\mathsf{v}_0^{3;\pm}| \le \dfrac{\sqrt{4/3}\,\sqrt{\langle \mathsf{V}_1^\pm | \mathsf{V}_1^\pm\rangle^\pm}}{\lambda_3^\pm}. \end{cases} \tag{C.8}$$

From (C.4) and (C.5), we have the following L^2-estimates:

$$\begin{aligned} &\int_{\boldsymbol{R}} \left\langle \mathsf{V}_0^\pm | \boldsymbol{P}_0^\pm(\xi^1 - s)\mathsf{V}_{0x}^\pm + \boldsymbol{P}_0^\pm(\xi^1 - s)\mathsf{V}_{1x}^\pm \right\rangle^\pm e^{\pm\frac{4}{5}\mathsf{e}\epsilon x} dx \\ &\quad + \int_{\boldsymbol{R}} \left\langle \mathsf{V}_1^\pm | \boldsymbol{P}_1^\pm(\xi^1 - s)\mathsf{V}_{0x}^\pm + \boldsymbol{P}_1^\pm(\xi^1 - s)\mathsf{V}_{1x}^\pm - L^\pm \mathsf{V}_1^\pm - \mathsf{R}^\pm\mathsf{V} - Q(\mathsf{V},\mathsf{V}) - \mathsf{S} \right\rangle^\pm \\ &\quad \times e^{\pm\frac{4}{5}\mathsf{e}\epsilon x} dx = 0. \end{aligned} \tag{C.9}$$

Here, the inner product $\langle g|h\rangle^\pm$ are defined as in Sect. 7:

$$\langle g|h\rangle^\pm \equiv \int_{\boldsymbol{R}^3} \frac{g(\xi)h(\xi)}{\omega_\pm(\xi)}\, d\xi.$$

From (C.8) and (C.9),

$$\int_{\boldsymbol{R}} \left\langle \mathsf{V}_0^{\pm} | \boldsymbol{P}_0^{\pm}(\xi^1 - s)\mathsf{V}_{0x}^{\pm} + \boldsymbol{P}_0^{\pm}(\xi^1 - s)\mathsf{V}_{1x}^{\pm} \right\rangle^{\pm} e^{\pm\frac{4}{5}\mathsf{e}\epsilon x} dx$$
$$+ \int_{\boldsymbol{R}} \left\langle \mathsf{V}_1^{\pm} | \boldsymbol{P}_1^{\pm}(\xi^1 - s)\mathsf{V}_{0x}^{\pm} + \boldsymbol{P}_1^{\pm}(\xi^1 - s)\mathsf{V}_{1x}^{\pm} \right\rangle^{\pm} e^{\pm\frac{4}{5}\mathsf{e}\epsilon x} dx$$
$$= \pm\frac{4}{5}\mathsf{e}\epsilon \int_{\boldsymbol{R}} \left(\frac{1}{2} \left\langle \mathsf{V}_0^{\pm} | (\xi^1 - s)\mathsf{V}_0^{\pm} \right\rangle^{\pm} - \left\langle \mathsf{V}_1^{\pm} | (\xi^1 - s)\mathsf{V}_1^{\pm} \right\rangle^{\pm} \right) e^{\pm\frac{4}{5}\mathsf{e}\epsilon x}\, dx$$
$$= \pm \int_{\boldsymbol{R}} \left(\sum_{j=1}^{3} \frac{2\mathsf{e}\epsilon}{5} \lambda_j^{\pm} (\mathsf{v}_0^{j;\pm})^2 - \frac{4}{5}\mathsf{e}\epsilon \left\langle \mathsf{V}_1^{\pm} | (\xi^1 - s)\mathsf{V}_1^{\pm} \right\rangle^{\pm} \right) e^{\pm\frac{4}{5}\mathsf{e}\epsilon x}\, dx$$
$$\leq \int_{\boldsymbol{R}} \frac{8\mathsf{e}\epsilon}{15} \frac{|\mathsf{V}_1^{\pm}|^2}{|\lambda_3^{\pm}|} e^{\pm\frac{4}{5}\mathsf{e}\epsilon x}\, dx + O(1)\,\epsilon \int_{\boldsymbol{R}} \left(|\mathsf{V}_1^{\pm}|^2 + \left\langle \mathsf{V}_1^{\pm} | (\xi^1 - s)\mathsf{V}_1^{\pm} \right\rangle^{\pm} \right) e^{\pm\frac{4}{5}\mathsf{e}\epsilon x}\, dx. \tag{C.10}$$

From (C.10) and (B.4),

$$\int_{\boldsymbol{R}} \left\langle \mathsf{V}_0^{\pm} | \boldsymbol{P}_0^{\pm}(\xi^1 - s)\mathsf{V}_{0x}^{\pm} + \boldsymbol{P}_0^{\pm}(\xi^1 - s)\mathsf{V}_{1x}^{\pm} \right\rangle^{\pm} e^{\pm\frac{4}{5}\mathsf{e}\epsilon x} dx$$
$$+ \int_{\boldsymbol{R}} \left\langle \mathsf{V}_1^{\pm} | \boldsymbol{P}_1^{\pm}(\xi^1 - s)\mathsf{V}_{0x}^{\pm} + \boldsymbol{P}_1^{\pm}(\xi^1 - s)\mathsf{V}_{1x}^{\pm} \right\rangle^{\pm} e^{\pm\frac{4}{5}\mathsf{e}\epsilon x} dx$$
$$\leq \frac{\nu_0}{12} \int_{\boldsymbol{R}} \langle \mathsf{V}_1^{\pm} | \mathsf{V}_1^{\pm} \rangle^{\pm} e^{\pm\frac{4}{5}\mathsf{e}\epsilon x}\, dx + O(1)\,\epsilon \int_{\boldsymbol{R}} \left(\langle \mathsf{V}_1^{\pm} | \mathsf{V}_1^{\pm} \rangle^{\pm} + \left\langle \mathsf{V}_1^{\pm} | (\xi^1 - s)\mathsf{V}_1^{\pm} \right\rangle^{\pm} \right)$$
$$\times e^{\pm\frac{4}{5}\mathsf{e}\epsilon x}\, dx. \tag{C.11}$$

From (B.1), there exists $\mathsf{C} > 0$,

$$\left\langle \xi^1 \mathsf{V}_1^{\pm} | \mathsf{V}_1^{\pm} \right\rangle^{\pm} \leq -\mathsf{C} \langle \mathsf{V}_1^{\pm} | L^{\pm} \mathsf{V}_1^{\pm} \rangle^{\pm}. \tag{C.12}$$

And so

$$O(1)\epsilon \int_{\boldsymbol{R}} \left\langle (\xi^1 - s)\mathsf{V}_1^{\pm} | \mathsf{V}_1^{\pm} \right\rangle^{\pm} e^{\pm\mathsf{e}\epsilon x}\, dx \leq O(1)\epsilon \int_{\boldsymbol{R}} - \langle \mathsf{V}_1^{\pm} | L^{\pm} \mathsf{V}_1^{\pm} \rangle^{\pm} e^{\pm\mathsf{e}\epsilon x}\, dx. \tag{C.13}$$

From (C.3), (C.11), (C.12), and (C.13) together with Schwartz's inequality, for ϵ sufficiently small,

$$\int_{\boldsymbol{R}} \frac{\nu_0}{2} \quad \langle \mathsf{V}_1^{\pm} | \mathsf{V}_1^{\pm} \rangle^{\pm} e^{\pm\frac{4}{5}\mathsf{e}\epsilon x}\, dx \leq \int_{\boldsymbol{R}} -\frac{1}{2} \langle \mathsf{V}_1^{\pm} | L^{\pm} \mathsf{V}_1^{\pm} \rangle e^{\pm\frac{4}{5}\mathsf{e}\epsilon x}\, dx$$
$$\leq \int_{\boldsymbol{R}} \left\langle \mathsf{V}_0^{\pm} | \boldsymbol{P}_0^{\pm}(\xi^1 - s)\mathsf{V}_{0x}^{\pm} + \boldsymbol{P}_0^{\pm}(\xi^1 - s)\mathsf{V}_{1x}^{\pm} \right\rangle^{\pm} e^{\pm\frac{4}{5}\mathsf{e}\epsilon x} dx$$
$$+ \int_{\boldsymbol{R}} \left\langle \mathsf{V}_1^{\pm} | \boldsymbol{P}_1^{\pm}(\xi^1 - s)\mathsf{V}_{0x}^{\pm} + \boldsymbol{P}_1^{\pm}(\xi^1 - s)\mathsf{V}_{1x}^{\pm} - L^{\pm}\mathsf{V}_1^{\pm} - \mathsf{R}^{\pm}\mathsf{V} \right\rangle^{\pm} e^{\pm\frac{4}{5}\mathsf{e}\epsilon x} dx$$
$$\leq \int_{\boldsymbol{R}} \left(\frac{\nu_0}{8} \langle \mathsf{V}_1^{\pm} | \mathsf{V}_1^{\pm} \rangle^{\pm} + \frac{8}{\nu_0} \langle \mathsf{S} | \mathsf{S} \rangle^{\pm} + \langle \mathsf{V}_1^{\pm} | Q(\mathsf{V}, \mathsf{V}) \rangle^{\pm} \right) e^{\pm\frac{4}{5}\mathsf{e}\epsilon x}\, dx.$$

And so

$$\int_{\boldsymbol{R}} \frac{\nu_0}{4} \left\langle \mathsf{V}_1^{\pm} | \mathsf{V}_1^{\pm} \right\rangle^{\pm} e^{\pm \frac{4}{5} \mathsf{e} \epsilon x} \, dx \leq \int_{\boldsymbol{R}} -\frac{1}{4} \left\langle \mathsf{V}_1^{\pm} | L^{\pm} \mathsf{V}_1^{\pm} \right\rangle^{\pm} e^{\pm \frac{4}{5} \mathsf{e} \epsilon x} \, dx$$
$$\leq \int_{\boldsymbol{R}} \left(\frac{8}{\nu_0} \langle \mathsf{S} | \mathsf{S} \rangle^{\pm} + \left\langle \mathsf{V}_1^{\pm} | Q(\mathsf{V}, \mathsf{V}) \right\rangle^{\pm} \right) e^{\pm \frac{4}{5} \mathsf{e} \epsilon x} \, dx. \tag{C.14}$$

From (A.15a) and (A.15b),

$$\int_{\boldsymbol{R}} \langle \mathsf{S} | \mathsf{S} \rangle^{\pm} e^{\pm \frac{4}{5} \mathsf{e} \epsilon x} \, dx \leq O(1) \, \epsilon^5. \tag{C.15}$$

With (C.14) and (C.15), we make an *a priori* assumption,

$$\| \mathsf{V}_1^{\pm} \omega_{\pm}^{-\frac{1}{2}} \|_{\infty} \leq O(1) \, \epsilon^2. \tag{C.16}$$

Substitute (C.16) and (C.15) into (C.14). We have

$$\int_{\boldsymbol{R}} \frac{\nu_0}{8} \left\langle \mathsf{V}_1^{\pm} | \mathsf{V}_1^{\pm} \right\rangle^{\pm} e^{\pm \frac{4}{5} \mathsf{e} \epsilon x} \, dx \leq \int_{\boldsymbol{R}} -\frac{1}{8} \left\langle \mathsf{V}_1^{\pm} | L^{\pm} \mathsf{V}_1^{\pm} \right\rangle^{\pm} e^{\pm \frac{4}{5} \mathsf{e} \epsilon x} \, dx$$
$$\leq \int_{\boldsymbol{R}} \frac{8}{\nu_0} \langle \mathsf{S} | \mathsf{S} \rangle^{\pm} \; e^{\pm \frac{4}{5} \mathsf{e} \epsilon x} \, dx \leq O(1) \, \epsilon^5. \tag{C.17}$$

To obtain the desired higher order estimates in Theorem 8.1 and close the energy estimates, we increase the differentiation order in the *a priori* estimates in (C.16) up to C^{10} in ξ-x space and perform the energy estimate up to H^{20}. This proves the existence of the shock profile with the property in Theorem 8.1. The details are similar to the energy estimates before, so it is omitted.

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