A Fourth Order Magnus Scheme for Helmholtz Equation

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Abstract

For wave propagation in a slowly varying waveguide, it is necessary to solve the Helmholtz equation in a domain that is much larger than the typical wavelength. Standard finite difference and finite element methods must resolve the small oscillatory behavior of the wave field and are prohibitively expensive for practical applications. A popular method is to approximate the waveguide by segments that are uniform in the propagation direction and use separation of variables in each segment. For a slowly varying waveguide, it is possible that the length of such a segment is much larger than the typical wavelength. To reduce memory requirements, it is advantageous to reformulate the boundary value problem of the Helmholtz equation as an initial value problem using a pair of operators. Such an operator-marching scheme can also be solved with the piecewise uniform approximation of the waveguide. This is related to the second order midpoint exponential method for a system of linear ODEs. In this paper, we develop a fourth order operator-marching scheme for the Helmholtz equation using a fourth order Magnus method.

1 Introduction

For time-harmonic acoustic waves and transversely polarized electro-magnetic waves in a two-dimensional wave-guiding structure, the governing equation is the Helmholtz equation:

$$u_{xx} + u_{zz} + \kappa^2(x,z)u = 0, \tag{1}$$

where $\kappa = k_0 n(x,z)$, k_0 is the wavenumber in free space, n is the refractive index. The equation is considered in a parallel plane waveguide given by $0 < z < 1$ and $-\infty < x < +\infty$ and it is solved subject to the following boundary conditions:

$$u(x,0) = 0, \quad u_z(x,1) = 0. \tag{2}$$

We further assume that the waveguide is x-independent for $x < 0$ and $x > L$. The problem can then be reduced to the finite interval $0 \le x \le L$ with boundary conditions specified at $x = 0$ and $x = L$.

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In practical applications, the length scale of the domain is much larger than the typical wavelength. That is,

$$L >> 1 >> \lambda = \frac{2\pi}{k_0},$$

where λ is the free space wavelength. Standard finite difference [1, 2, 3] and finite element [4, 5, 6] methods need to use a sufficient number of grid points for each wavelength to resolve the small-scale oscillatory behavior of the wave field. Meanwhile, the grid size (say, Δ) must decrease when k_0 increases. In fact, because of the so-called pollution effect [6], an even smaller value of $k_0\Delta$ is needed for a larger k_0. These methods give rise to very large linear systems that are difficult to solve. On the other hand, we are often interested in slowly varying waveguides, where $n(x, z)$ changes with x slowly. Efficient numerical schemes can be developed to take advantage of this special feature. In fact, when n is x-independent, the equation can be solved by separation of variables based on the eigenfunctions of the transverse operator $\partial_z^2 + \kappa^2$. For a slowly varying waveguide, the popular coupled mode method [7, 8, 9] approximates the waveguide by x-independent segments and use separation of variables in each segment. Consider a discretization

$$0 = x_0 < x_1 < x_2 < \ldots < x_m = L,$$

the coupled mode method [7] approximates $n(x, z)$ by $n(x_{j+1/2}, z)$ for $x_j < x < x_{j+1}$, where $x_{j+1/2} = (x_j + x_{j+1})/2$. This is a second order approximation and the overall error is expected to be $O(h^2)$, where $h = x_{j+1} - x_j$. However, unlike the error produced by a second order finite difference method [1], the error of the coupled mode method vanishes when the waveguide is x-independent. Therefore, the method is useful for slowly varying waveguides.

In this paper, we develop a fourth order method based on a Magnus integrator [10] for linear evolution equations. The error is expected to be $O(h^4)$ [11] and it vanishes when n is x-independent. Numerical examples involving slowly varying waveguides are used to illustrate the accuracy and the large stepping capacity of the method.

2 Operator-marching methods

In the coupled mode method [7], the wave field in the interval (x_j, x_{j+1}) is written as

$$u(x, z) = \sum_{k=1}^{K} \left[c_{k,j} e^{i\beta_{k,j}(x - x_{j+1/2})} + d_{k,j} e^{-i\beta_{k,j}(x - x_{j+1/2})} \right] \phi_{k,j}(z)$$

where $\phi_{k,j}$ and $\beta_{k,j}^2$ are the eigenfunctions and eigenvalues of the local transverse operator $\partial_z^2 + \kappa^2(x_{j+1/2}, z)$, satisfying

$$\frac{d^2\phi}{dz^2} + \kappa^2(x_{j+1/2}, z)\phi = \beta^2\phi$$

and $\phi(0) = \phi'(1) = 0$. The subscript j is used to indicate that the eigenvalues and eigenfunctions are specific for the interval (x_j, x_{j+1}) and the index k is used to order the eigenvalues as a decreasing sequence. The positive and negative eigenvalues give real and pure imaginary $\beta_{k,j}$, and they correspond to the propagating and evanescent modes, respectively. Notice that the expansion for u is truncated to K terms. Typically, K is chosen to be slightly larger than the number of propagating modes in every interval. The evanescent modes decay exponentially in the x-direction and they are not very important in a slowly varying waveguide. For a general κ, the first K eigenfunctions can only be calculated numerically. When a finite difference or a finite element method is used to approximate the above eigenvalue problem and z is discretized by N points, a pair of eigenvalue and eigenfunction can be found in $O(N)$ operations by Rayleigh quotient iteration. Since K eigenpairs are needed for each interval, the total number of operations needed for solving the local eigenvalue problems is $O(mKN)$. The coupled mode method also needs to evaluate the overlap matrix related to

$$\int_0^1 \phi_{k,j}(z)\phi_{s,j-1}(z)dz.$$

These are $K \times K$ matrices related to the coupling of modes in nearby segments. Each entry of an overlap matrix can be calculated in $O(N)$ operations, since the eigenfunctions are represented by vectors of length N. The total number of operations for computing all overlap matrices is thus $O(mK^2N)$. Finally, the coefficients $c_{k,j}$ and $d_{k,j}$ are solved from the continuity conditions of u and u_x at x_0, x_1, ..., x_m. The required number of operations is $O(mK^3)$ and the required memory space is $O(mK^2)$.

It is possible to reduce the memory requirement to $O(K^2)$ while keeping the number of operations unchanged, using a one-way reformulation of the Helmholtz equation in terms of a pair of operators. One approach is to use the reflection and transmission operators [12]. A simpler approach is to use the Dirichlet-to-Neumann (DtN) map and the fundamental solution (FS) operator [13]. Unlike the scattering operators, the DtN and FS operators are continuous at the discontinuities x_j $(j = 0, 1, ..., m)$ and their manipulations are more convenient. The operators are represented by $K \times K$ matrices based on the local eigenfunction expansions. The operators are solved as an initial value problem from $x = L$ to $x = 0$. In the step from x_j to x_{j-1}, it is necessary to manipulate $K \times K$ matrices representing the operators and $O(K^3)$ operations are required. The total required number of operations is also $O(mK^3)$. As in the coupled mode method, additional work is needed to calculate the K eigenfunctions in each interval and the $K \times K$ overlap matrices.

To describe the DtN-FS reformulation, we first consider the boundary conditions at $x = 0$ and $x = L$. For $x < 0$ and $x > L$, the refractive index $n(x, z)$ is x-independent. Let

$$\kappa(x, z) = \kappa_0(z) \quad \text{for} \quad x < 0,$$
$$\kappa(x, z) = \kappa_\infty(z) \quad \text{for} \quad x > L.$$

For $x > L$, we assume that there are only outgoing waves propagating towards $x = +\infty$ and evanescent waves that decays in the increasing x direction. This gives rise to the radiation condition

$$u_x = i\sqrt{\partial_z^2 + \kappa_\infty^2}\; u \qquad (3)$$

at $x = L$. Originally, the above is valid for $x > L$, but it is also valid at $x = L$, since u and u_x are continuous. For $x < 0$, the wave field can be decomposed as $u = u^{(i)} + u^{(r)}$ for the incident and reflected waves. The incident wave is assumed to be given and the reflected wave is to be determined. They satisfy

$$\begin{aligned} u_x^{(i)} &= i\sqrt{\partial_z^2 + \kappa_0^2}\; u^{(i)} \\ u_x^{(r)} &= -i\sqrt{\partial_z^2 + \kappa_0^2}\; u^{(r)}. \end{aligned}$$

This gives rise to the boundary condition

$$u_x + i\sqrt{\partial_z^2 + \kappa_0^2}\; u = 2i\sqrt{\partial_z^2 + \kappa_0^2}\; u^{(i)} \tag{4}$$

at $x = 0$.

For each x, we define the DtN map $Q(x)$ and FS operator $Y(x)$ by

$$Q(x)u(x,z) = u_x(x,z), \tag{5}$$

$$Y(x)u(x,z) = u(L,z) \tag{6}$$

for all u satisfying the Helmholtz equation, the boundary conditions at $z = 0$, $z = 1$ and the radiation condition at $x = L$. Then, the operators satisfy [13]

$$Q' = -Q^2 - [\partial_z^2 + \kappa^2(x,z)], \tag{7}$$

$$Y' = -YQ(x), \tag{8}$$

and they should be solved with the initial conditions

$$Q(L) = i\sqrt{\partial_z^2 + \kappa_\infty^2}, \tag{9}$$

$$Y(L) = 1. \tag{10}$$

The idea is to solve Q and Y from $x = L$ to $x = 0$. From $Q(0)$ and $Y(0)$, we can then construct the reflected wave $u^{(r)}$ (at $x = 0-$) and the transmitted wave $u(L,z)$.

3 Magnus integrator

The initial value problem for Q and Y (when represented as matrices in a local eigenfunction expansion) can be solved by standard methods for ODE initial value problems or more efficient methods [13] specially developed for the Riccati equation. However, these methods are all based on difference approximations in the x-direction and require a small step size even when the waveguide is x-independent. To develop a more efficient method for slowly varying waveguide, we could also use the piecewise uniform waveguide approximation as in the coupled mode method. For the interval $x_j < x < x_{j+1}$, we approximate $n(x,z)$ by $n(x_{j+1/2}, z)$, then find the exact relationships [14] between (Q_{j+1}, Y_{j+1}) and (Q_j, Y_j). This leads to a second order method which uses about the same amount of operations as the coupled mode method but less computer memory.

It turns out that this second order operator-marching method [14] is related to the midpoint exponential method

$$y_{j+1} = e^{hA(x_{j+1/2})}\, y_j$$

for the linear system

$$\frac{dy}{dx} = A(x)\, y, \tag{11}$$

where

$$y = \begin{bmatrix} u_x \\ u \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -\partial_z^2 - \kappa^2 \\ 1 & 0 \end{bmatrix}. \tag{12}$$

Therefore, it is natural to develop more accurate operator-marching schemes based on more accurate integrators of the linear evolution equation. Notice that for a general linear system, when A has an eigenvalue with a large imaginary part, the solution is highly oscillatory. Thus, standard methods for ODE initial value problems require a small step size. On the other hand, the midpoint exponential method is exact if A does not depend on x, thus a relatively larger h can be used when A depends on x slowly. We derive a fourth order method for solving Q and Y based on the following fourth order Magnus method [10, 11]:

$$y_{j+1} = e^{\Omega_j}\, y_j \quad \text{for} \quad \Omega_j = \frac{h}{2}(A_1 + A_2) + \frac{\sqrt{3}h^2}{12}(A_2 A_1 - A_1 A_2), \tag{13}$$

where

$$A_k = A(x_j + c_k h) \quad \text{for} \quad k = 1, 2, \quad c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}, \quad c_1 = \frac{1}{2} + \frac{\sqrt{3}}{6}.$$

Notice that when A is x-independent, the method is again exact.

For y and A given in (12) and for A_1 and A_2 given above, we have

$$A_2 A_1 - A_1 A_2 = \begin{bmatrix} \kappa_1^2 - \kappa_2^2 & 0 \\ 0 & \kappa_2^2 - \kappa_1^2 \end{bmatrix},$$

where $\kappa_1 = \kappa(x_j + c_1 h, z)$ and $\kappa_2 = \kappa(x_j + c_2 h, z)$. Therefore,

$$\Omega_j = h \begin{bmatrix} d_1 & -\partial_z^2 - d_0 \\ 1 & -d_1 \end{bmatrix},$$

where

$$d_0 = \frac{1}{2}(\kappa_1^2 + \kappa_2^2), \quad d_1 = \frac{\sqrt{3}h}{12}(\kappa_1^2 - \kappa_2^2).$$

To evaluate e^{Ω_j}, we need to diagonalize the matrix. For this purpose, we consider the eigenvalue problem

$$\begin{bmatrix} d_1 & -\partial_z^2 - d_0 \\ 1 & -d_1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \Lambda \\ X_2 \Lambda \end{bmatrix},$$

where Λ is the operator eigenvalue. This gives rise to

$$(-\partial_z^2 - d_0 + d_1^2) X_2 = X_2 \Lambda^2.$$

We can choose $X_2 = 1$, then $\Lambda = \pm iB$ and $X_1 = \Lambda + d_1$, where

$$B = \sqrt{\partial_z^2 + d_0 - d_1^2} = \sqrt{\partial_z^2 + \frac{1}{2}(\kappa_1^2 + \kappa_2^2) - \frac{h^2}{48}(\kappa_1^2 - \kappa_2^2)^2}. \tag{14}$$

Therefore,

$$\begin{bmatrix} d_1 & -\partial_z^2 - d_0 \\ 1 & -d_1 \end{bmatrix} = S \begin{bmatrix} iB & 0 \\ 0 & -iB \end{bmatrix} S^{-1} \quad \text{for} \quad S = \begin{bmatrix} iB + d_1 & -iB + d_1 \\ 1 & 1 \end{bmatrix}.$$

Let $v = u_x$, the step from x_j to x_{j+1} can be evaluated as

$$\begin{bmatrix} v_{j+1} \\ u_{j+1} \end{bmatrix} = e^{\Omega_j} \begin{bmatrix} v_j \\ u_j \end{bmatrix} = S \begin{bmatrix} e^{ihB} & 0 \\ 0 & e^{-ihB} \end{bmatrix} S^{-1} \begin{bmatrix} v_j \\ u_j \end{bmatrix}.$$

or

$$S^{-1} \begin{bmatrix} v_{j+1} \\ u_{j+1} \end{bmatrix} = \begin{bmatrix} e^{ihB} & 0 \\ 0 & e^{-ihB} \end{bmatrix} S^{-1} \begin{bmatrix} v_j \\ u_j \end{bmatrix}.$$

It turns out that

$$S^{-1} = \frac{(iB)^{-1}}{2} \begin{bmatrix} 1 & iB - d_1 \\ -1 & iB + d_1 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} e^{ihB}(iB + Q_j - d_1)u_j &= (iB + Q_{j+1} - d_1)u_{j+1} \\ (iB - Q_j + d_1)u_j &= e^{ihB}(iB - Q_{j+1} + d_1)u_{j+1}. \end{aligned}$$

Here, we have replaced v by Qu following Eq. (5) and $Q_j \approx Q(x_j)$, etc. If we eliminate u_{j+1}, we obtain a formula between Q_j and Q_{j+1}. It can be written as

$$P = (iB - Q_{j+1} + d_1)(iB + Q_{j+1} - d_1)^{-1}, \tag{15}$$

$$R = e^{ihB} P e^{ihB}, \tag{16}$$

$$Q_j = d_1 + (1 + R)^{-1}(1 - R)iB. \tag{17}$$

From the definition of Y in (6), we have

$$Y_j u_j = Y_{j+1} u_{j+1} = u(L, z).$$

This gives rises to

$$Y_j = Y_{j+1}(iB + Q_{j+1} - d_1)^{-1} e^{ihB}(iB + Q_j - d_1).$$

The above can be simplified to

$$Y_j = Y_{j+1}(iB)^{-1}(1 + P)e^{ihB}(1 + R)^{-1} iB. \tag{18}$$

It is possible to further simplify the formulas. Let

$$\tilde{P} = (iB + Q_{j+1} - d_1)^{-1}(iB - Q_{j+1} + d_1), \tag{19}$$

$$\tilde{R} = e^{ihB} \tilde{P} e^{ihB}. \tag{20}$$

We have

$$P = iB\tilde{P}(iB)^{-1}, \quad R = iB\tilde{R}(iB)^{-1},$$

and then

$$Q_j = d_1 + iB(1+\tilde{R})^{-1}(1-\tilde{R}), \tag{21}$$

$$Y_j = Y_{j+1}(1+\tilde{P})e^{ihB}(1+\tilde{R})^{-1}. \tag{22}$$

In summary, the equations for Q and Y are solved backwards from $x = L$ to $x = 0$. For the step from x_{j+1} to x_j, Q_{j+1} and Y_{j+1} are given, we then move to the next step through B, $\tilde{P}$ and $\tilde{R}$ defined in (14),(19) and (20), respectively, and evaluate Q_j and Y_j by (21) and (22).

Here the algorithm is given in terms of the operators. In practice, Q and Y are represented by $K \times K$ matrices through truncated expansions in the eigenfunctions of the quasi-local operator B^2 in (14), i.e. $\partial_z^2 + (\kappa_1^2 + \kappa_2^2)/2 - h^2(\kappa_1^2 - \kappa_2^2)^2/48$. The integer K is the number of retained eigenfunctions. The details are similar to the method described in Ref. [13].

4 Numerical examples

As an example, we consider a waveguide where

$$\kappa^2(x,z) = k_0^2 \left[1 + \epsilon\, e^{-20(x/L-0.5)^2} \sin^2(\pi z)\right]. \tag{23}$$

We start with some calculations for

$$L = 10, \quad k_0 = 10, \quad \epsilon = 0.05.$$

We discretize z by $N = 30$ points, namely, $z_j = j/(N+0.5)$ for $j = 1, 2, ..., N$. A fourth order finite difference scheme is used to approximate the operator ∂_z^2. The homogeneous waveguide at infinity, i.e, $\kappa = k_0$, supports 3 propagating modes. It is enough to truncate the local eigenfunction expansion to six terms, that is, $K = 6$. Our method is used to calculate the solution at $x = L$ based on the following Dirichlet boundary condition at $x = 0$:

$$u(0,z) = \sum_{j=1}^{7} \sin(m_j z_0)\sin(m_j z)/\sqrt{k_0^2 - m_j^2} \quad \text{for} \;\; m_j = (j - 1/2)\pi, \quad z_0 = 0.65. \tag{24}$$

A reference solution is first calculated with $h = 1/256$. After that, larger values of the step size h are used and the corresponding numerical solutions are compared with the reference solution to calculate the relative errors (denoted by $E(h)$) in the L^2 norm. The relative errors for $h = 2, 1, 1/2, 1/4, ..., 1/64$ are shown in Fig. 1(a), using a logarithmic scale. It gives a clear indication that the method is indeed fourth order. The large stepping capacity is also quite clear. In fact, as it is illustrated in Fig. 1(b), the result obtained with $h = 2$ is already quite accurate.

We also calculate the back-scattered wave $u^{(r)}$ at $x = 0$ generated by the following incident wave:

$$u^{(i)}(0,z) = \sin(2.5\pi z).$$

This corresponds to the third propagating mode in the waveguide (away from the distortion near $x = L/2$). Since the waveguide has a very gradual variation in the x direction, the back-scattered wave is quite weak. However, we are able to obtain a fairly accurate solution with

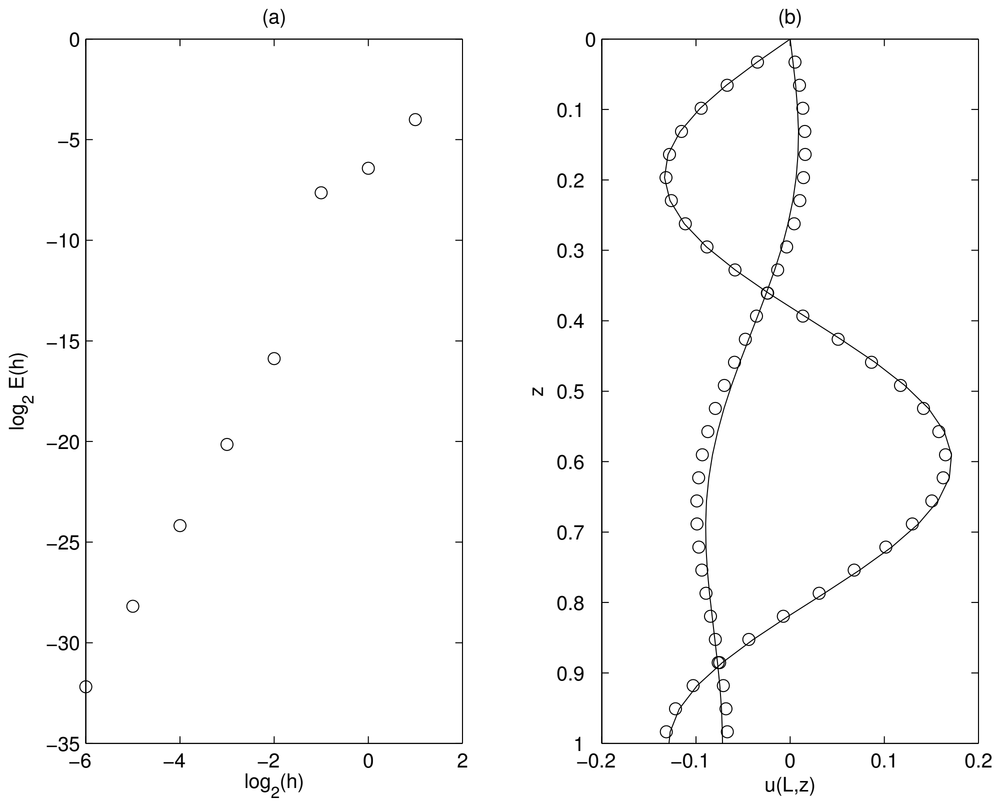

Figure 1: (a). Relative errors of $u(L,z)$ for various values of h; (b). Comparison of $u(L,z)$ for $h=2$ (little circles) and $h=1/256$ (solid lines).

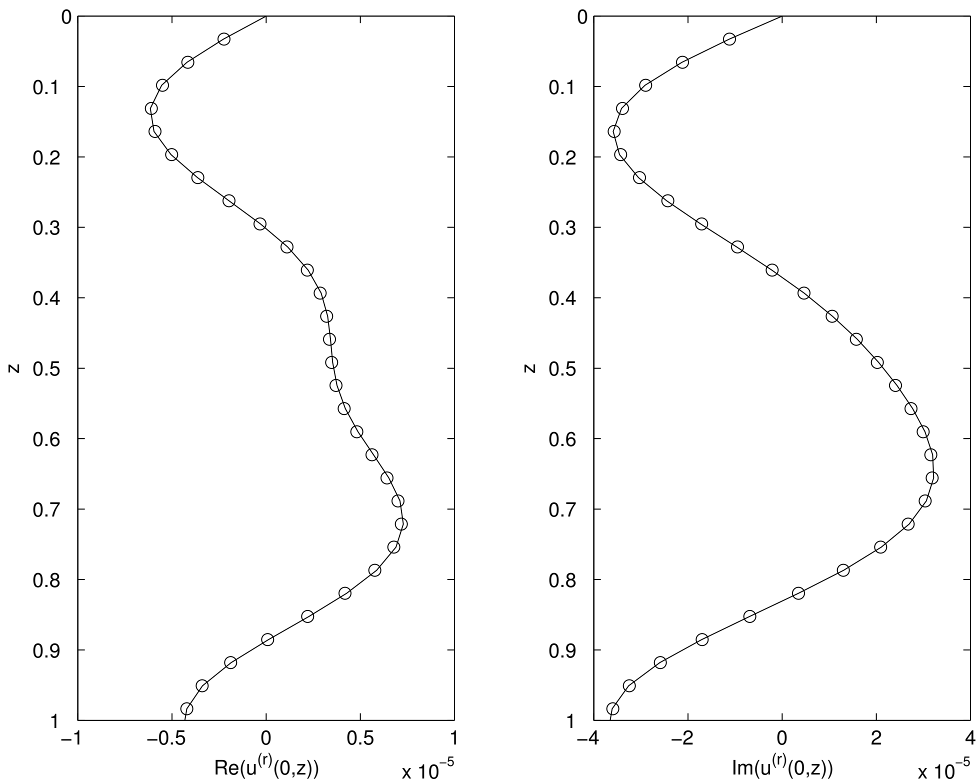

Figure 2: A comparison of the back-scattered waves calculated with step size $h = 1/8$ (small circles) and $h = 1/256$ (solid lines).

$h = 1/8$. In Fig. 2, the numerical solution for $u^{(r)}(0, z)$ obtained with $h = 1/8$ is compared with a much more accurate solution calculated with $h = 1/256$. Notice that the magnitude is only around 10^{-5}. Using the numerical solution obtained with $h = 1/256$ as the "exact" solution, we compute the relative errors in the L^2 norm for a few values of h. The results are listed in Table 1.

h	1/8	1/16	1/32	1/64
$E(h)$	8.74×10^{-3}	5.16×10^{-4}	3.18×10^{-5}	1.97×10^{-6}

Table 1: Relative errors of $u^{(r)}(0, z)$ in the L^2 norm.

Our method is designed for waveguides that vary with x slowly. The parameter ϵ in (23) is an indicator for the x-dependence of the structure. At the fixed step size $h = 1/4$, we calculate the relative error of $u(L, z)$ for various values of ϵ. The relative error is computed using a reference solution obtained with $h = 1/256$. The other parameters are fixed at $L = 10$, $k_0 = 10$, $N = 30$ and $K = 6$. The results are plotted in Fig. 3. It is clear that the numerical solution for

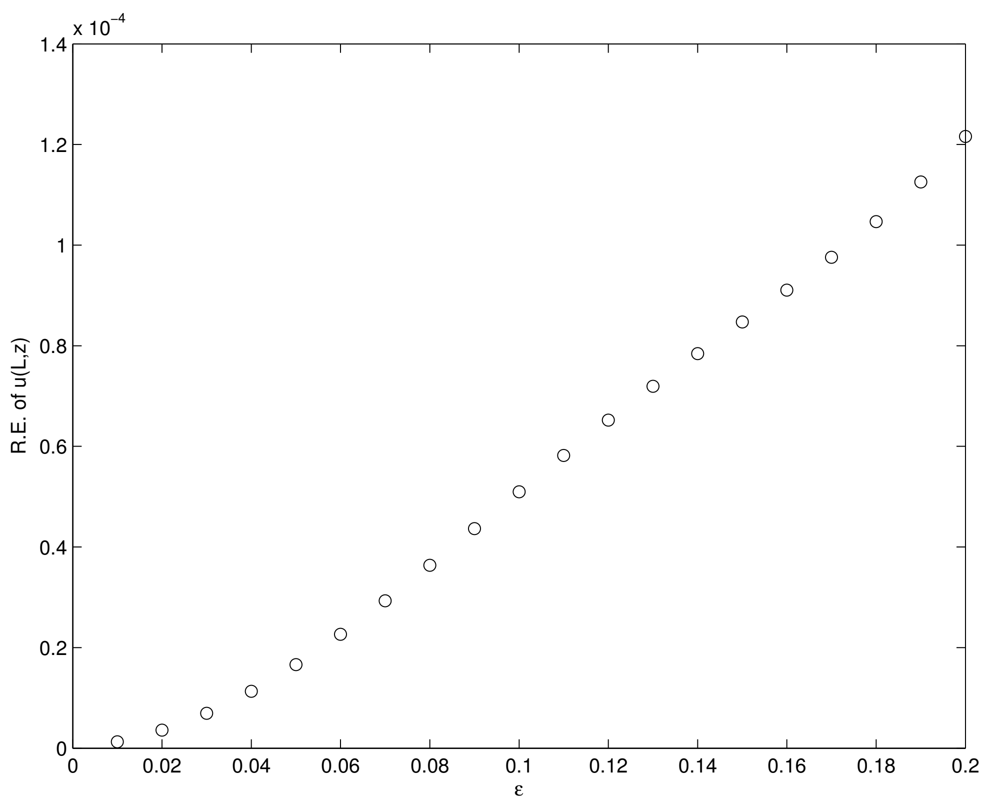

Figure 3: The dependence of relative error of $u(L, z)$ on ϵ for the fixed step size $h = 1/4$.

the fixed h becomes more accurate, when ϵ is decreased. A more interesting case is when the x-dependence is weakened, but the total length L is increased simultaneously. This is a highly relevant limit, because a typical waveguide may vary slowly with x, but it could have a very large length scale in the propagation direction. For the profile given in (23), if we increase L

and keep ϵ fixed, the x-dependence is already weakened, because the magnitude of $\partial\kappa(x,z)/\partial x$ decreases as L is increased. In the following, we choose $\epsilon = 0.2$, $k_0 = 10$ and $h = 1/4$, and calculate the relative error of $u(L,z)$ for $10 \le L \le 80$. As before, we take $N = 30$, $K = 6$ and calculate a reference solution with $h = 1/256$. The results are shown in Fig. 4. We observe

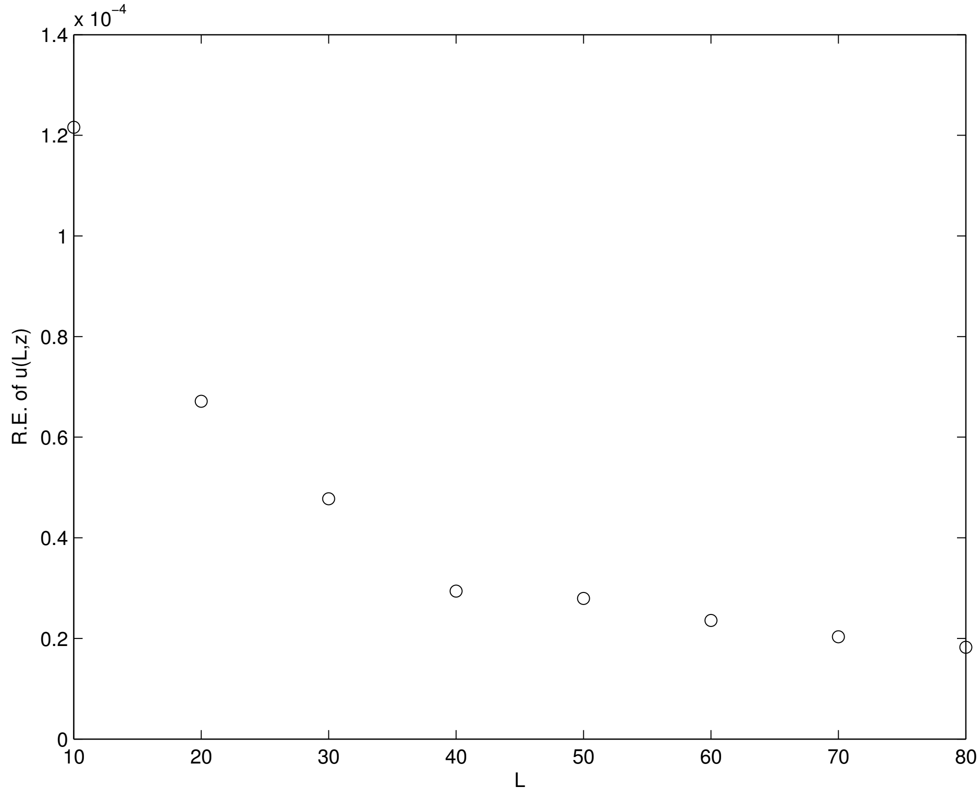

Figure 4: The relative error of $u(L,z)$ calculated with $h = 1/4$ as a function of L.

that the solution becomes more accurate when L is increased.

Finally, we consider the problem of increasing k_0. At a higher frequency, the wave field becomes more oscillatory, standard numerical methods must use a sufficient number of grid points in each wavelength. In fact, because of the pollution effect, it is necessary to use even more points per wavelength for a larger value of k_0. In our method, the variable z is discretized by N points using a finite difference method to calculate the first K eigenfunctions of a transverse operator. Clearly, the number N must also provide a sufficient number of points in each wavelength. Meanwhile, when k_0 is increased, the number of propagating modes increases, thus K must also be increased. In the following numerical experiment, we choose $\epsilon = 0.05$, $L = 10$ and consider $k_0 = 10, 20, ..., 80$. The integers N and K are chosen to increase linearly with k_0, namely, from 30 to 240 for N and from 6 to 48 for K. The step size h is chosen to satisfy $hk_0 = 5/2$. For each k_0, we calculate the relative error of $u(L,z)$ by comparing the numerical solution with a more accurate solution obtained by replacing h with $h/16$. The same N and K are used to calculate the reference solution, since they appear to be large enough. The results shown in Fig. 5 reveal that the solution in general becomes more accurate when k_0 is increased,

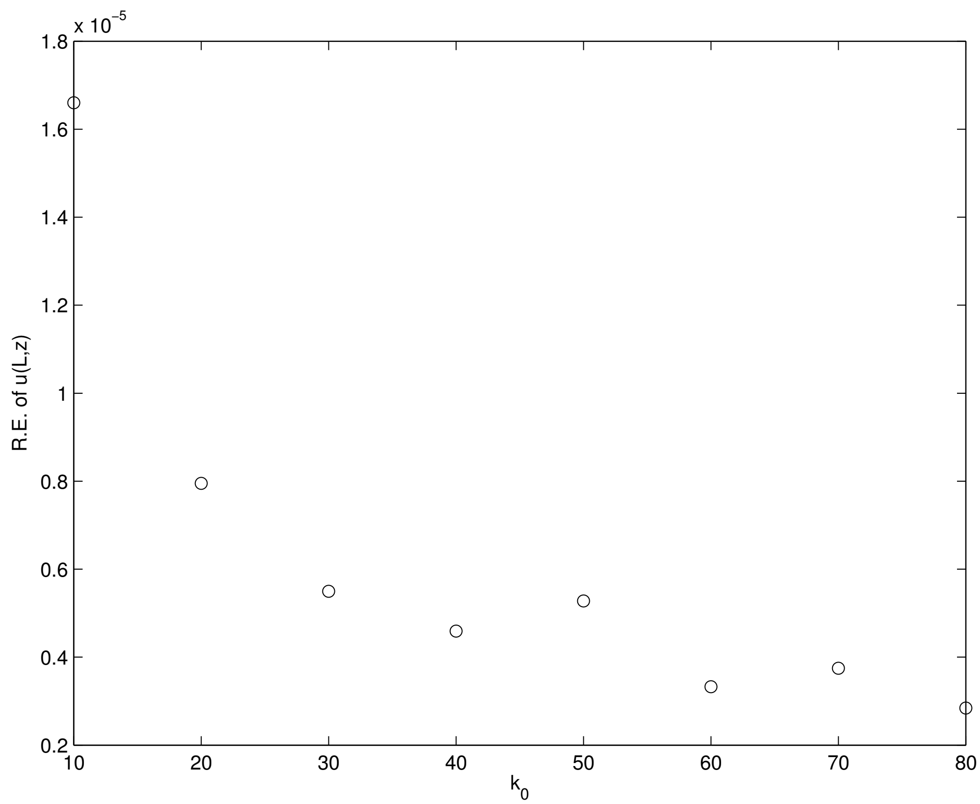

Figure 5: Relative error of $u(L, z)$ for different values of k_0 while hk_0 is fixed at 5/2.

while hk_0 is fixed at $5/2$. Although the relative error does not decrease with k_0 monotonically, a clear tendency of decreasing is observed when k_0 is doubled from $k_0 = 10$ to 20, 40 and 80. In any event, this is in contrast to the pollution effect observed in standard finite difference and finite element methods.

5 Conclusions

A numerical method is developed for the Helmholtz equation in a slowly varying waveguide. The method is derived from the fourth order Magnus method [10] for linear evolution equations. In our implementation, a one-way re-formulation of the Helmholtz equation in terms of the Dirichlet-to-Neumann map is used. In the propagation direction x, the method has a fourth order of accuracy and it also preserves the exact solution when the waveguide is x-independent. In general, the step size in x is not severely restricted by the wavelength of the field and it can be relatively large, when the x-dependence is weak. Although the method has similar performance compared with the fourth order method developed in [14], no derivatives of $\kappa(x, z)$ are needed. For waveguide with curved boundaries, it is necessary to first flatten the waveguide by a transformation to avoid the crude "star-case" approximation [15]. In that case, the coefficients of the Helmholtz equation are rather complicated and their derivatives are difficult to evaluate. The derivative-free operator marching method developed in this paper can be advantageous in this case.

Numerical examples are used to demonstrate the the fourth order accuracy of the method and also the good accuracy obtained with larger step sizes in x when the dependence on x is weak. From the numerical experiments, it appears that when the total length of the waveguide is increased and the x-dependence of the refractive index profile is decreased accordingly, the numerical solution at a fixed step size h becomes more accurate. Similarly, we observe a general tendency that the solution becomes more accurate when k_0 is increased, if k_0h is kept as fixed. This is consistent with the error estimate in [11] for the fourth order Magnus method applied to the Schrödinger equation. Nevertheless, for a fixed h, the numerical solution becomes less accurate as k_0 is increased. Recently, a number of powerful numerical methods [17, 18, 19] are developed for linear evolution equations with highly oscillatory solutions. The possibility of applying these new techniques to the Helmholtz equation will be explored in future works.

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