City University of Hong Kong

Course code & title:	MA6606 Computational Linear Algebra
Session:	Semester A, 2004-2005
Date:	16 December 2004
Time:	18:30 — 21:30
Time allowed:	Three hours

This paper has THREE pages. (Including this page)

Instructions to candidates:

- Attempt all **SEVEN** questions.
- Start each question on a new page.
- Show all working.

Materials, aids & instruments permitted to be used during examination:

- Non-programmable portable battery operated calculator.

1. (15 marks) Find the 2-norm of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

2. (15 marks) Find the LU decomposition with *partial pivoting* for

$$A = \begin{bmatrix} -3/2 & -1 & 0 \\ 3 & 2 & 6 \\ -1 & 4/3 & -6 \end{bmatrix}.$$

3. (15 marks) Let A be a 5×3 matrix with full column rank (i.e. $rank(A) = 3$) and assume that A has a QR factorization: $A = QR$, where

$$Q = I - \frac{2}{v^*v} vv^*, \quad v = [1, 0, -1, 0, 1]^*.$$

For the column vector $b = [1, 2, 3, 2, 1]^*$, find

$$\min_{x \in C^3} ||Ax - b||.$$

4. (15 marks) Consider the matrix-vector multiplication $y = Ax$, where

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Let a, b, c, x_1 and x_2 be floating point numbers, show that $\tilde{y}$ — the computer result of y, satisfies

$$\tilde{y} = \begin{bmatrix} a & \tilde{b} \\ 0 & c \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix},$$

for some $\tilde{b}$, $\tilde{x}_1$ and $\tilde{x}_2$ which are close to b, x_1 and x_2, respectively.

5. (15 marks) For the following symmetric matrix

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 1 & 0 \end{bmatrix},$$

find a Householder reflection H, such that $T = HAH^*$ is symmetric tridiagonal. Calculate the matrix T.

6. (15 marks) Let r_0, r_1 and r_2 be the three columns of the 3×3 indentity matrix. Determine three vectors p_0, p_1 and p_2, such that

$$< p_0 >=< r_0 >, \; < p_0, p_1 >=< r_0, r_1 >, \; < p_0, p_1, p_2 >=< r_0, r_1, r_2 >$$

and $p_j^* A p_k = 0$ whenever $j \neq k$, where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Here, $< p_0, p_1 >$ is the vector space spanned by p_0 and p_1, etc.

7. Let A be a real symmetric matrix, b be a real unit vector which is not an eigenvector of A. With $q_1 = b$, we use Lanczos method to obtain

$$A\,[q_1, q_2, ...] = [q_1, q_2, ...] \begin{bmatrix} \alpha_1 & \beta_1 & \\ \beta_1 & \alpha_2 & \ddots \\ & \ddots & \ddots \end{bmatrix}.$$

(a) (2 marks) Show that b and Ab are linearly independent.

(b) (2 marks) Show that q_1 and q_2 are linearly independent.

(c) (2 marks) Show that $\mathcal{K}_2 =< b, Ab >=< q_1, q_2 >$.

(d) (4 marks) Let s_1 be the smallest eigenvalue of $T_2 = \begin{bmatrix} \alpha_1 & \beta_1 \\ \beta_1 & \alpha_2 \end{bmatrix}$, show that

$$s_1 = \min_{x \in \mathcal{K}_2} \frac{x^* A x}{x^* x}.$$

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