City University of Hong Kong

Course code & title:	MA6606 Computational Linear Algebra
Session:	Semester B, 2000-2001
Time allowed:	Three hours

This paper has FOUR pages. (Including this page)

Instructions to candidates:

- The paper has **seven** questions.
- Attempt only **SIX** questions.
- All questions carry equal marks.
- Start each question on a new page.
- Show all working.

Materials, aids & instruments permitted to be used during examination:

- Non-programmable portable battery operated calculator.

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1. Calculate the matrix 2-norm for

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}.$$

2. Let A be an $m \times m$ real symmetric matrix ($m > 3$). Assume that A has three known eigenvalues:

$$\lambda_1 = 4, \quad \lambda_2 = 2, \quad \lambda_3 = 1.$$

Let v_1, v_2 and v_3 be the unit eigenvectors corresponding to λ_1, λ_2 and λ_3, respectively, and

$$b = v_1 + v_2 + 3v_3.$$

Let $p(\lambda) = \lambda^3 + z_1\lambda^2 + z_2\lambda + z_3$ be a monic polynomial of degree 3. Formulate the minimization problem

$$\min_{z_1,z_2,z_3 \in \Re} ||p(A)b||_2$$

as a standard least squares problem. That is, find a matrix B, a vector d, such that

$$||p(A)b||_2 = ||Bz - d||_2,$$

where z is the column vector with components z_1, z_2 and z_3. You do not need to solve this least squares problem.

3. Consider the problem of solving x_1, x_2 and x_3 from

$$\begin{bmatrix} a_1 & 1 & 0 \\ & a_2 & 1 \\ & & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

where a_1, a_2 and a_3 are non-zero real numbers. Describe an algorithm for calculating x_1, x_2 and x_3. If a_1, a_2 and a_3 are non-zero floating point numbers and floating point operations are used in your algorithm, show that the computer results, say $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{x}_3$, satisfy

$$\begin{bmatrix} \tilde{a}_1 & 1 & 0 \\ & \tilde{a}_2 & 1 \\ & & \tilde{a}_3 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

for some $\tilde{a}_1$, $\tilde{a}_2$ and $\tilde{a}_3$ close to a_1, a_2 and a_3, respectively.

4. Let A be a real symmetric matrix given by

$$A = \begin{bmatrix} a & b & c^* \\ b & d & e^* \\ c & e & F \end{bmatrix},$$

where a, $b \neq 0$ and d are scalars, c and e are column vectors, F is a symmetric matrix. Find a unit lower triangular matrix L, such that

$$LAL^* = \begin{bmatrix} a & b & 0 \\ b & d & \tilde{e}^* \\ 0 & \tilde{e} & \tilde{F} \end{bmatrix}.$$

Give formulas for $\tilde{e}$ and $\tilde{F}$.

5. Let A be a real 3×3 singular (i.e. $\det(A) = 0$) matrix having the following structure:

$$A = \begin{bmatrix} * & 1 & 0 \\ 1 & * & 1 \\ 0 & 1 & * \end{bmatrix},$$

where $*$ denotes some possibly non-zero entry. Let $A = QR$ be the QR factorization of A, show that $\hat{A} = RQ$ has the following structure:

$$\hat{A} = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

6. Let A and b be given as

$$A = \begin{bmatrix} 2 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Calculate the matrix Q_2 and T_2 (such that $AQ_2 \approx Q_2T_2$) by the Lanczos method starting from $q_1 = b/||b||_2$, where $Q_2 = [q_1, q_2]$ is the matrix of first two Lanczos vectors, T_2 is 2×2 and symmetric.

7. For the matrix A and vector b below, we define the function

$$\phi(x) = \frac{1}{2}x^* Ax - x^* b,$$

where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and x is a real column vector of length 3. Solve the minimization problem

$$\min_{\alpha_1, \alpha_2 \in \Re} \phi(\alpha_1 p_0 + \alpha_2 p_1),$$

where p_0 and p_1 are given by

$$p_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad p_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}.$$

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