MA3514 Mid-term Test (18/03/2009, 2:30pm–4:20pm)

1. (10 marks) For the following ODE initial value problem:

$$y'' + (1+2t)y = 0, \quad t > 0,$$
$$y(0) = 1, \quad y'(0) = 0,$$

find the approximate values of $y(0.1)$ and $y'(0.1)$, based on the implicit midpoint method and a step size $h = 0.1$.

2. Consider the following numerical method for ODE initial value problem:

$$y_{j+1} + \frac{1}{4}y_j - \frac{1}{2}y_{j-1} - \frac{3}{4}y_{j-2} = \frac{h}{8}\left(19f_j + 5f_{j-1}\right),$$

where the ODE is $y' = f(t, y)$ and $f_j = f(t_j, y_j)$, etc.

   (a) (5 marks) Find its local truncation error and the order.

   (b) (5 marks) Analyze its zero-stability.

3. (10 marks) The shooting method reformulates the following ODE boundary value problem

$$y'' + y = x(y')^2, \quad 0 < x < 1,$$
$$y(0) = 1, \quad y'(1) = y'(0),$$

as a nonlinear equation $\phi(t) = 0$. Define the variable t and the function ϕ. How can $\phi(t)$ and $\phi'(t)$ be calculated numerically for a **given** t?

4. (a) (2 marks) Let a be a non-zero real constant and $i = \sqrt{-1}$, find the analytic solution of $y' = iay$ for $t > 0$, with initial condition $y(0) = 1$.

   (b) (5 marks) If the midpoint method is applied to the simple ODE in (a), show that $|y_j| \to \infty$ as $j \to \infty$, where the step size h is positive and fixed.

   (c) (3 marks) A physics student was trying to solve the one-dimensional Schrödinger equation for a quantum mechanics class:

$$i\hbar\, u_t = -\frac{\hbar^2}{2m}u_{xx} + V(x)u,$$

   where $i = \sqrt{-1}$, $\hbar$ is a real constant (reduced Planck's constant), m is the mass of a particle (e.g. an electron), $V(x)$ is a given function (the potential energy), u the "wave function" of the particle (related to the probability distribution of the particle). He first discretized u_{xx} by the central difference approximation (so the equation is approximated by an ODE system), then applied the midpoint method. Do you think he will be able to get accurate results when the time step h is small enough? Why?