MA3514 Exam Solutions
(Semester B, 2008-2009)

1. We first have to re-write the 2nd order differential equation as a first order system:

$$\frac{d}{dt}\vec{y} = \frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \vec{f}(t,\vec{y}) = \begin{bmatrix} y' \\ t + y - 2y' \end{bmatrix}$$

where $\vec{y} = \begin{bmatrix} y \\ y' \end{bmatrix}$.

Now, for BDF2, we have

$$\begin{bmatrix} y_{j+1} \\ y'_{j+1} \end{bmatrix} - \frac{4}{3}\begin{bmatrix} y_j \\ y'_j \end{bmatrix} + \frac{1}{3}\begin{bmatrix} y_{j-1} \\ y'_{j-1} \end{bmatrix} = \frac{2h}{3}\begin{bmatrix} y'_{j+1} \\ t_{j+1} + y_{j+1} - 2y'_{j+1} \end{bmatrix}$$

for $h = 0.15$, $\quad 2h/3 = 0.1 \qquad t_{j+1} = 0.15(j+1)$

$$\Rightarrow \begin{bmatrix} 1 & -0.1 \\ -0.1 & 1.2 \end{bmatrix}\begin{bmatrix} y_{j+1} \\ y'_{j+1} \end{bmatrix} = \frac{4}{3}\begin{bmatrix} y_j \\ y'_j \end{bmatrix} - \frac{1}{3}\begin{bmatrix} y_{j-1} \\ y'_{j-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0.015(j+1) \end{bmatrix}$$

This gives rise to

$$A = \frac{4}{3}\begin{bmatrix} 1 & -0.1 \\ -0.1 & 1.2 \end{bmatrix}^{-1} \approx \begin{bmatrix} 1.3445 & 0.1120 \\ 0.1120 & 1.1204 \end{bmatrix}$$

$$B = -\frac{1}{3}\begin{bmatrix} 1 & -0.1 \\ -0.1 & 1.2 \end{bmatrix}^{-1} \approx -\begin{bmatrix} 0.3361 & 0.0280 \\ 0.0280 & 0.2801 \end{bmatrix}$$

$$\vec{c} = \begin{bmatrix} 1 & -0.1 \\ -0.1 & 1.2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0.015(j+1) \end{bmatrix} \approx \begin{bmatrix} 0.0012605 \\ 0.012605 \end{bmatrix}(j+1).$$

2. The method is for $y' = f(t, y)$.

(a). If for $y' = 0$, then $y_{j+1} - y_{j-1} = 0$

the general solution of this linear recurrence relation is: $y_j = A \cdot 1^j + B(-1)^j = A + B(-1)^j$

Since y_j is bounded for all j, the method is zero-stable.

(b) for $A(0)$-stable, we consider

$y' = ay$, where a is real and $a < 0$.

The method gives:

$$y_{j+1} - y_{j-1} = \frac{ha}{3}(y_{j+1} + 4y_j + y_{j-1}).$$

Let $s = \frac{ah}{3}$, then:

$$(1-s)y_{j+1} - 4sy_j - (1+s)y_{j-1} = 0$$

The general solution is:

$$y_j = A\lambda_1^j + B\lambda_2^j.$$

where λ_1, λ_2 satisfy

$$(1-s)\lambda^2 - 4s\lambda - (1+s) = 0.$$

For A(0)-stable, we must have $y_j \to 0$ as $j \to \infty$ for any $a<0$ and any $h>0$. That means

$|\lambda_1| < 1$ and $|\lambda_2| < 1$ for all $s<0$.

However, from the quadiatic equation of λ, we have $\lambda_1 + \lambda_2 = \frac{4s}{1-s}$

Therefore, $|\lambda_1 + \lambda_2| = \frac{4|s|}{|1-s|} > 2$ if $|s|$ is large enough. But the requirement $|\lambda_1| < 1$ and $|\lambda_2| < 1$ implies that

$$|\lambda_1 + \lambda_2| \le |\lambda_1| + |\lambda_2| < 2.$$

Therefore, the method cannot be A(0)-stable.

3. (a). We have these Taylor expansions:

$$y(x+h) = y(x) + h y'(x) + \frac{h^2}{2} y''(x) + \frac{h^3}{3!} y'''(x) + \frac{h^4}{4!} y^{(4)}(x) + \frac{h^5}{5!} y^{(5)}(x) + \frac{h^6}{6!} y^{(6)}(x) + \cdots$$

$$y(x-h) = y(x) - h y'(x) + \frac{h^2}{2} y''(x) - \frac{h^3}{3!} y'''(x) + \frac{h^4}{4!} y^{(4)}(x) - \frac{h^5}{5!} y^{(5)}(x) + \frac{h^6}{6!} y^{(6)}(x) + \cdots$$

$$y''(x+h) = y''(x) + h y'''(x) + \frac{h^2}{2} y^{(4)}(x) + \frac{h^3}{3!} y^{(5)}(x) + \frac{h^4}{4!} y^{(6)}(x) + \cdots$$

$$y''(x-h) = y''(x) - h y'''(x) + \frac{h^2}{2} y^{(4)}(x) - \frac{h^3}{3!} y^{(5)}(x) + \frac{h^4}{4!} y^{(6)}(x) + \cdots$$

thus: $y(x+h)+y(x-h) = 2y(x) + h^2 y''(x) + \frac{h^4}{12} y^{(4)}(x) + \frac{h^6}{360} y^{(6)}(x) + \cdots$

$$y''(x+h)+y''(x-h) = 2y''(x) + h^2 y^{(4)}(x) + \frac{h^4}{12} y^{(6)}(x) + \cdots$$

Left hand side of (1)

$$= \frac{1}{h^2}\left[h^2 y''(x) + \frac{h^4}{12} y^{(4)}(x) + \frac{h^6}{360} y^{(6)}(x) + \cdots\right]$$

$$- \frac{1}{12}\left[12 y''(x) + h^2 y^{(4)}(x) + \frac{h^4}{12} y^{(6)}(x) + \cdots\right]$$

$$= \left(\frac{1}{360} - \frac{1}{12^2}\right) h^4 y^{(6)}(x) + \cdots = -\frac{1}{240} h^4 y^{(6)}(x) + \cdots = O(h^4).$$

(b). We discretize x by

$$x_j = jh, \quad h = \frac{1}{n+1}.$$

and let $y_j \approx y(x_j)$. then

$$\frac{1}{h^2}\left[y_{j-1} - 2y_j + y_{j+1}\right] \approx \frac{1}{12}\left[y''(x_{j-1}) + 10 y''(x_j) + y''(x_{j+1})\right]$$

Insert $y''(x_j) = r(x_j) - y(x_j) \approx r(x_j) - y_j$,

etc into the above, we get

$$y_{j-1} - 2y_j + y_{j+1} = \frac{h^2}{12}\left[r_{j-1} - y_{j-1} + 10(r_j - y_j) + r_{j+1} - y_{j+1}\right]$$

$$\Rightarrow \left(1+\frac{h^2}{12}\right) y_{j-1} + \left(-2 + \frac{5h^2}{6}\right) y_j + \left(1 + \frac{h^2}{12}\right) y_{j+1} = \frac{h^2}{12}\left(r_{j-1} + 10 r_j + r_{j+1}\right)$$

where $r_j = r(x_j)$, etc.

The above equation is valid for $j = 1, 2, \cdots, n$.

We also have the boundary conditions.

$$y_0 = y_{n+1} = 0.$$

4. (a) If we discretize x by x_j and t by t_k with grid size Δx and time step Δt, and denote the numerical solution $u_j^k \approx u(x_j, t_k)$, then we can discretize as

$$\frac{1}{(\Delta t)^2}\left[u_j^{k+1} - 2u_j^k + u_j^{k-1}\right] = \frac{1}{(\Delta x)^2}\left[u_{j-1}^k - 2u_j^k + u_{j+1}^k\right] + c(x_j)\, u_j^k$$

(b). For the stability analysis, we let

$$u_j^k = \rho^k e^{i\beta x_j}, \text{ then}$$

$$\frac{1}{(\Delta t)^2}\left[\rho - 2 + \frac{1}{\rho}\right] = \frac{1}{(\Delta x)^2}\left[e^{i\beta\cdot\Delta x} - 2 + e^{-i\beta\cdot\Delta x}\right] + c$$

$$\Rightarrow \rho + \frac{1}{\rho} = 2 + \left(\frac{\Delta t}{\Delta x}\right)^2\left[2\cos(\beta\Delta x) - 2\right] + c\,(\Delta t)^2$$

$$\Rightarrow \quad \rho + \frac{1}{\rho} = 2 + \left(\frac{\Delta t}{\Delta x}\right)^2 (-4) \sin^2\left(\frac{\beta \Delta x}{2}\right) + C \cdot (\Delta t)^2$$

$$= 2r$$

where $r = 1 - 2\left(\frac{\Delta t}{\Delta x}\right)^2 \sin^2\left(\frac{\beta \Delta x}{2}\right) + \frac{C}{2}(\Delta t)^2$

$$\Rightarrow \quad \rho^2 - 2r\rho + 1 = 0$$

$$\rho = \frac{2r \pm \sqrt{4r^2 - 4}}{2} = r \pm \sqrt{r^2 - 1}$$

If $|r| > 1 \Rightarrow$ one of ρ satisfies $|\rho| > 1$

$\Rightarrow$ unstable.

If $|r| \leq 1 \Rightarrow \rho = r \pm \sqrt{1 - r^2}\, i$

$|\rho| = 1 \Rightarrow$ stable.

For stability, we must have $|r| \leq 1$ for all β.

If $C > 0$, we choose $\beta = 0$, then $r = 1 + \frac{C}{2}(\Delta t)^2 > 1$

$\Rightarrow$ unstable.

If $C < 0$, we choose $\underbrace{\frac{\beta \Delta x}{2} = \frac{\pi}{2}}_{\text{to have a minimum of } r}$, then require

$$r = 1 - 2\left(\frac{\Delta t}{\Delta x}\right)^2 + \frac{C}{2}(\Delta t)^2 \geq -1.$$ This gives

$$\Delta t \leq \frac{\Delta x}{\sqrt{1 - \frac{C}{4}(\Delta x)^2}}.$$

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In conclusion, if $c>0$, the method is unconditionally unstable; if $c<0$, the method is conditionally stable and the stability condition is

$$\Delta t \leq \frac{\Delta x}{\sqrt{1-\frac{c}{4}(\Delta x)^2}}.$$

5. The 2nd order finite difference scheme gives:

$$W_{ij} - \frac{\Delta t}{2h^2}\left(W_{i,j-1} - 2W_{ij} + W_{i,j+1}\right)$$

$$= U_{ij}^{k} + \frac{\Delta t}{2h^2}\left(U_{i,j-1}^{k} - 2U_{ij}^{k} + U_{i,j+1}^{k}\right)$$

Let $T = \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}$ and using the zero boundary condition, we obtain:

$$W\left(I - \frac{\Delta t}{2h^2}T\right) = U^{k}\left(I + \frac{\Delta t}{2h^2}T\right).$$

Similarly, the equation $\left(1 - s\partial_x^2\right)U^{k+1} = \left(1 + s\partial_x^2\right)w$

gives:

$$\left(I - \frac{\Delta t}{2h^2}T\right)U^{k+1} = \left(I + \frac{\Delta t}{2h^2}T\right)W$$

Therefore:

$$U^{k+1} = \left(I - \frac{\Delta t}{2h^2} T\right)^{-1} \left(I + \frac{\Delta t}{2h^2} T\right) U^k \left(I + \frac{\Delta t}{2h^2} T\right) \left(I - \frac{\Delta t}{2h^2} T\right)^{-1}.$$

6. (a). For $\frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right)$, we discretize it by:

$$\frac{1}{\Delta x}\left[p(x_{i+\frac{1}{2}}) \frac{1}{\Delta x}(u_{i+1,j} - u_{ij}) - p(x_{i-\frac{1}{2}}) \frac{1}{\Delta x}(u_{ij} - u_{i-1,j})\right]$$

$$= \frac{1}{(\Delta x)^2}\left[c_i (u_{i+1,j} - u_{ij}) + a_i (u_{i-1,j} - u_{ij})\right]$$

where $a_i = p(x_{i-\frac{1}{2}})$

$c_i = p(x_{i+\frac{1}{2}})$

$x_{i-\frac{1}{2}} = (i - \frac{1}{2})\Delta x$ $\qquad$ $x_{i+\frac{1}{2}} = (i + \frac{1}{2})\Delta x$

$$\Delta x = h = \frac{1}{n+1}$$

Meanwhile, $\frac{\partial^2 u}{\partial y^2}$ is approximated by

$$\frac{1}{h^2}\left[u_{i,j-1} - 2u_{ij} + u_{i,j+1}\right]$$

Therefore: $a_i = p(x_{i-\frac{1}{2}})$, $c_i = p(x_{i+\frac{1}{2}})$,

$$f_{ij} = h^2 F(x_i, y_j).$$

(b). Let $f_{ij} = \sum_{k=1}^{n} \hat{f}_{ik} \sin \frac{jk\pi}{n+1}$, $1 \le j \le n$

$u_{ij} = \sum_{k=1}^{n} \hat{u}_{ik} \sin \frac{jk\pi}{n+1}$, $1 \le j \le n$.

then:

$$\sum_k \left\{ a_i \hat{u}_{i-1,k} + c_i \hat{u}_{i+1,k} - (2 + a_i + c_i) \hat{u}_{ik} \right\} \sin \frac{jk\pi}{n+1}$$
$$+ \sum_k \hat{u}_{ik} \left[\sin \frac{(j-1)k\pi}{n+1} + \sin \frac{(j+1)k\pi}{n+1} \right] = \sum_k \hat{f}_{ik} \sin \frac{jk\pi}{n+1}$$

$$\sin \frac{(j-1)k\pi}{n+1} + \sin \frac{(j+1)k\pi}{n+1} = 2 \cos \frac{k\pi}{n+1} \cdot \sin \frac{jk\pi}{n+1}$$

$$\Rightarrow \sum_k \sin \frac{jk\pi}{n+1} \left\{ a_i \hat{u}_{i-1,k} + c_i \hat{u}_{i+1,k} - \left(2 + a_i + c_i - 2 \cos \frac{k\pi}{n+1}\right) \hat{u}_{ik} - \hat{f}_{ik} \right\} = 0$$

$\Rightarrow$ Let $b_i^{(k)} = -2 - a_i - c_i + 2 \cos \frac{k\pi}{n+1}$

$\Rightarrow$ $a_i \hat{u}_{i-1,k} + b_i^{(k)} \hat{u}_{ik} + c_i \hat{u}_{i+1,k} = \hat{f}_{ik}$

From the zero boundary conditions we have

$\hat{u}_{0k} = 0 = \hat{u}_{n+1,k}$

$$\Rightarrow: \begin{bmatrix} b_1^{(k)} & c_1 & & \\ a_2 & b_2^{(k)} & c_2 & \\ & \ddots & \ddots & c_{n-1} \\ & & a_n & b_n^{(k)} \end{bmatrix} \begin{bmatrix} \hat{u}_{1k} \\ \hat{u}_{2k} \\ \vdots \\ \hat{u}_{nk} \end{bmatrix} = \begin{bmatrix} \hat{f}_{1k} \\ \hat{f}_{2k} \\ \vdots \\ \hat{f}_{nk} \end{bmatrix}$$

This is the k-th system involving only n unknowns.