City University of Hong Kong

Course code & title:	MA3514 Numerical Methods for Differential Equations
Session:	Semester B, 2008-2009
Date:	16 May 2009
Time:	14:00–17:00
Time allowed:	Three hours

This paper has **three** pages. (Including this page)

Instructions to candidates:

- Attempt **all** questions.
- Start each question on a new page.
- Show all working.

Materials, aids & instruments permitted to be used during examination:

- Non-programmable portable battery operated calculator.

1. (13 marks) When the numerical method BDF2

$$\mathbf{y}_{j+1} - \frac{4}{3}\mathbf{y}_j + \frac{1}{3}\mathbf{y}_{j-1} = \frac{2h}{3}\mathbf{f}(t_{j+1}, \mathbf{y}_{j+1})$$

is applied to the differential equation

$$y'' + 2y' - y = t, \quad \text{for} \quad t > 0,$$

we obtain

$$\begin{bmatrix} y_{j+1} \\ y'_{j+1} \end{bmatrix} = A \begin{bmatrix} y_j \\ y'_j \end{bmatrix} + B \begin{bmatrix} y_{j-1} \\ y'_{j-1} \end{bmatrix} + \mathbf{c},$$

where A and B are matrices, $\mathbf{c}$ is a vector, $y_j \approx y(t_j)$, $y'_j \approx y'(t_j)$, $t_j = jh$ and h is the step size. Find A, B and $\mathbf{c}$ for $h = 0.15$.

2. Consider the Simpson's method for solving differential equations (which is related to Simpson's method for numerical integration):

$$y_{j+1} - y_{j-1} = \frac{h}{3}(f_{j+1} + 4f_j + f_{j-1}).$$

   (a) (5 marks) Show that the method is zero-stable.

   (b) (8 marks) Show that the method is not $A(0)$-stable.

3. (a) (7 marks) Let y be a function of x with continuous 6th order derivative, show that

$$\frac{y(x-h) - 2y(x) + y(x+h)}{h^2} - \frac{y''(x-h) + 10y''(x) + y''(x+h)}{12} = O(h^4). \qquad (1)$$

   (b) (7 marks) Describe a 4th order finite difference method based on Eq. (1), for

$$y'' + y = r(x), \quad 0 < x < 1,$$
$$y(0) = y(1) = 0,$$

   where r is a given function and it is sufficiently smooth.

4. (a) (8 marks) Propose an explicit second order finite difference method for

$$u_{tt} = u_{xx} + c(x)u,$$

   where $c(x)$ is a given function.

   (b) (12 marks) Analyze the stability of your numerical method for part (a), assuming that $c = c(x)$ is a constant.

5. (20 marks) Consider the heat equation $u_t = u_{xx} + u_{yy}$ on the unit square

$$S = \{(x, y) \mid 0 < x < 1,\ 0 < y < 1\},$$

with a zero boundary condition: $u = 0$ on ∂S (the boundary of S) for all t. The following method is proposed to calculate the numerical solution from t_k to $t_{k+1} = t_k + \Delta t$:

$$\begin{aligned} \left(1 - s\partial_y^2\right) w &= \left(1 + s\partial_y^2\right) u^k, \\ \left(1 - s\partial_x^2\right) u^{k+1} &= \left(1 + s\partial_x^2\right) w, \end{aligned}$$

where $s = \Delta t/2$, u^k denotes u at t_k, u^{k+1} denotes u at t_{k+1}, and w is a temporary function of x and y. In the fully discrete case, we discretize x and y by $x_i = ih$, $y_j = jh$ and $h = 1/(n+1)$ where n is a positive integer, and denote the numerical solution as

$$u_{ij}^k \approx u(x_i, y_j, t_k).$$

Let U^k and U^{k+1} be the $n \times n$ matrices whose (i, j) entries are u_{ij}^k and u_{ij}^{k+1}, respectively, express U^{k+1} in terms of U^k.

6. Let $p(x)$ and $F(x, y)$ be given functions, we consider the differential equation

$$\frac{\partial}{\partial x}\left[p(x)\frac{\partial u}{\partial x}\right] + \frac{\partial^2 u}{\partial y^2} = F(x, y),$$

on the unit square $S = \{(x, y) \mid 0 < x < 1,\ 0 < y < 1\}$, with a zero boundary condition: $u = 0$ on ∂S. For the discretization $x_i = ih$ and $y_j = jh$ where i, j and n are integers and $h = 1/(n+1)$, a finite difference method gives

$$a_i u_{i-1,j} + c_i u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - (2 + a_i + c_i)u_{ij} = f_{ij}, \quad \text{for} \quad 1 \le i,\ j \le n, \qquad (2)$$

where $u_{ij} \approx u(x_i, y_j)$ is the numerical solution.

(a) (8 marks) What are a_i, c_i and f_{ij}?

(b) (12 marks) Show that the system of equations (2) can be reduced to n systems by the discrete sine transform, where each system involves n unknowns.

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