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MA3514 Exam Solutions for 07-08.

1. Let $y = \begin{bmatrix} u \\ u' \\ u'' \end{bmatrix}$, then $y' = f(t, y) = \begin{bmatrix} u' \\ u'' \\ 1+t+u \end{bmatrix}$.

$$y_0 = \begin{bmatrix} u(0) \\ u'(0) \\ u''(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad t_0 = 0, \quad k_1 = f(t_0, y_0) = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}.$$

$h = 0.2$,

$$k_2 = f\left(t_0 + \frac{h}{2}, y_0 + \frac{h}{2} k_1\right) = f\left(0.1, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0.1 \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}\right)$$

$$= f\left(0.1, \begin{bmatrix} 1.2 \\ 2.3 \\ 3.2 \end{bmatrix}\right) = \begin{bmatrix} 2.3 \\ 3.2 \\ 1 + 0.1 + 1.2 \end{bmatrix} = \begin{bmatrix} 2.3 \\ 3.2 \\ 2.3 \end{bmatrix}$$

$$y_1 = y_0 + h k_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0.2 \begin{bmatrix} 2.3 \\ 3.2 \\ 2.3 \end{bmatrix} = \begin{bmatrix} 1.46 \\ 2.64 \\ 3.46 \end{bmatrix}.$$

Therefore $u(h) \approx 1.46$.

2. For $h = 0.25$ and $x_j = jh$, we have $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{3}{4}$, $g(x)$ is not continuous at x_2.

For $j=1$ and $j=3$, we have

$$\frac{1}{h^2}\left(u_{j-1} - 2u_j + u_{j+1}\right) + g(x_j)\, u_j = x_j$$

For $j=2$, we start with Taylor series:

$$u(x_3) \approx u(x_2) + h u'(x_2) + \frac{h^2}{2} u''(x_2^+)$$
$$u(x_1) \approx u(x_2) - h u'(x_2) + \frac{h^2}{2} u''(x_2^-)$$

$$\Rightarrow \quad u(x_3) \approx u(x_2) + h u'(x_2) + \frac{h^2}{2}\left[x_2 - g(x_2^+)u(x_2)\right]$$
$$u(x_1) \approx u(x_2) - h u'(x_2) + \frac{h^2}{2}\left[x_2 - g(x_2^-)u(x_2)\right]$$

eliminate $u'(x_2)$, we get:

$$u(x_1) + u(x_3) \approx 2u(x_2) + h^2\left(x_2 - \frac{1}{2}\left[g(x_2^+) + g(x_2^-)\right] u(x_2)\right]$$

$$\Rightarrow \frac{1}{h^2}\left[u_1 - 2u_2 + u_3\right] + \frac{1}{2}\left[g(x_2^-) + g(x_2^+)\right] u_2 = x_2.$$

Now: $g(x_1) = 1 + \frac{1}{4} = \frac{5}{4}$, $g(x_2^-) = 1 + \frac{1}{2} = \frac{3}{2}$, $g(x_2^+) = 1$, $g(x_3) = 1$

$$\Rightarrow \quad u_0 - 2u_1 + u_2 + \frac{5}{64} u_1 = \frac{1}{64} \Rightarrow u_0 - \frac{123}{64} u_1 + u_2 = \frac{1}{64}.$$

$$u_1 - 2u_2 + u_3 + \frac{5}{64} u_2 = \frac{1}{32} \Rightarrow u_1 - \frac{123}{64} u_2 + u_3 = \frac{1}{32}$$

$$u_2 - 2u_3 + u_4 + \frac{1}{16} u_3 = \frac{3}{64} \Rightarrow u_2 - \frac{31}{16} u_3 + u_4 = \frac{3}{64}$$

from the boundary conditions, we have $u_0 = u_4 = 0$.

Therefore;
$$\begin{bmatrix} -\frac{123}{64} & 1 & 0 \\ 1 & -\frac{123}{64} & 1 \\ 0 & 1 & -\frac{31}{16} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{64} \\ \frac{1}{32} \\ \frac{3}{64} \end{bmatrix}$$

3. For $u_t = u_{xx}$, the Crank-Nicolson method is

$$\frac{1}{\Delta t}\left[u_j^{k+1} - u_j^k\right] = \frac{1}{2(\Delta x)^2}\left[u_{j-1}^k - 2u_j^k + u_{j+1}^k + u_{j-1}^{k+1} - 2u_j^{k+1} + u_{j+1}^{k+1}\right].$$

Let $s = \frac{\Delta t}{2(\Delta x)^2}$, then

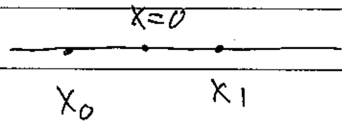
$$-s\,u_{j-1}^{k+1} + (1+2s)u_j^{k+1} - s\,u_{j+1}^{k+1} = s\,u_{j-1}^k + (1-2s)u_j^k + s\,u_{j+1}^k$$

From the boundary condition $u(1,t)=1$, we have

$$u_4^k = 1 \quad \text{for} \quad k = 0, 1, \cdots$$

From the boundary condition $u_x(0,t)=0$, we have

$$\frac{u_1^k - u_0^k}{\Delta x} = 0,$$



Therefore; $u_0^k = u_1^k$, $k = 0, 1, \cdots$

$$\Rightarrow \quad (1+s)u_1^{k+1} - s\,u_2^{k+1} = (1-s)u_1^k + s\,u_2^k.$$

Now $s = \dfrac{8/49}{2 \times \left(\frac{1}{3.5}\right)^2} = 1$, therefore

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s u_4^1 + s u_4^0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{7} \\ 3/7 \\ 5/7 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 3/7 \\ 12/7 \end{bmatrix},$$

$$u_j^0 = u(x_j, 0) = x_j = \left(j - \tfrac{1}{2}\right) \cdot \tfrac{2}{7}$$

4. First, we have the Taylor expansion

$$u(x, t+\Delta t) \approx u(x,t) + \Delta t \cdot u_t(x,t) + \frac{(\Delta t)^2}{2} u_{tt}(x,t)$$

Now, $\quad u_t = u^2 - u_x$

$$u_{tt} = 2u\,u_t - u_{tx} = 2u(u^2 - u_x) - (u^2 - u_x)_x$$

$$= 2u^3 - 2uu_x - 2uu_x + u_{xx}$$

$$= 2u^3 - 4uu_x + u_{xx}$$

Therefore,

$$u(x, t+\Delta t) \doteq u(x,t) + \Delta t \cdot (u^2 - u_x) + \frac{(\Delta t)^2}{2}\left[2u^3 - 4uu_x + u_{xx}\right]$$

$$u(x, t+\Delta t) \approx \left(u + (\Delta t)u^2 + (\Delta t)^2 u^3\right) - \left(\Delta t + 2(\Delta t)^2 u\right)u_x + \frac{(\Delta t)^2}{2} u_{xx}, \quad \text{where } u = u(x,t)$$

Next, we discretize above with a central difference approximation.

$$u_j^{k+1} = u_j^k + (\Delta t)\left(u_j^k\right)^2 + (\Delta t)^2\left(u_j^k\right)^3 - \frac{\Delta t}{2\Delta x}\left(1 + 2\Delta t \cdot u_j^k\right)\cdot\left(u_{j+1}^k - u_{j-1}^k\right) + \frac{(\Delta t)^2}{2(\Delta x)^2}\left(u_{j+1}^k - 2u_j^k + u_{j-1}^k\right).$$

5. let $u_j^k = \rho^k e^{i\beta x_j}$, then

$$\frac{1}{\Delta t}\left[\rho - \frac{1}{2}\left(e^{i\beta\cdot\Delta x} + e^{-i\beta\cdot\Delta x}\right)\right] = \frac{1}{2\Delta x}\left[e^{i\beta\Delta x} - e^{-i\beta\Delta x}\right]$$

$$\Rightarrow \rho - \cos(\beta\Delta x) = \frac{\Delta t}{2\Delta x}\cdot 2i\sin(\beta\Delta x)$$

$$= s\cdot i\cdot\sin(\beta\Delta x), \quad s = \frac{\Delta t}{\Delta x}$$

$$\Rightarrow |\rho|^2 = \cos^2(\beta\Delta x) + s^2\sin^2(\beta\Delta x)$$

If $s \le 1$ then $|\rho| \le 1$, the method is stable

If $s > 1$, we can choose β s.t.

$\sin^2(\beta\Delta x) = 1$ then $|\rho| > 1$, thus the method is unstable.

Thus, the stability condition is

$$\frac{\Delta t}{\Delta x} \le 1.$$

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6. With $u_0 = 0$ and $u_{n+1} = 0$, we have

$$u_{j-1} + a u_j + u_{j+1} = f_j$$

Now
$$u_j = \sum \hat{u}_k \sin \frac{jk\pi}{n+1}$$
$$f_j = \sum \hat{f}_k \sin \frac{jk\pi}{n+1}$$

$$\Rightarrow \sum_{k=1}^{n} \left[\hat{u}_k \sin \frac{(j-1)k\pi}{n+1} + a \hat{u}_k \sin \frac{jk\pi}{n+1} + \hat{u}_k \sin \frac{(j+1)k\pi}{n+1} \right]$$

$$= \sum_{k=1}^{n} \hat{u}_k \left(a + 2\cos \frac{k\pi}{n+1} \right) \sin \frac{jk\pi}{n+1}$$

$$= \sum_{k=1}^{n} \hat{f}_k \sin \frac{jk\pi}{n+1}$$

$$\Rightarrow \quad \hat{u}_k = \frac{\hat{f}_k}{a + 2\cos \frac{k\pi}{n+1}}.$$

If we take: $\hat{u} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ — l-th element

$\Rightarrow$

$$u_j = \sin \frac{jl\pi}{n+1}$$ ⇐ eigenvector for l-th eigenvalue.

and $$\hat{f} = \left[a + 2\cos \frac{l\pi}{n+1} \right] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad f_j = \left(a + 2\cos \frac{l\pi}{n+1} \right) \sin \frac{jl\pi}{n+1}$$ $\Rightarrow$ eigenvalue $= a + 2\cos \frac{l\pi}{n+1}$

for $l = 1, 2, \cdots, n$