City University of Hong Kong

Course code & title:	MA3514 Numerical Methods for Differential Equations
Session:	Semester B, 2007-2008
Time allowed:	Three hours

This paper has **three** pages. (Including this page)

Instructions to candidates:

- Attempt **all** questions.
- Start each question on a new page.
- Show all working.

Materials, aids & instruments permitted to be used during examination:

- Non-programmable portable battery operated calculator.

1. (20 marks) For the following ODE initial value problem

$$
\begin{aligned}
&u''' = 1 + t + u, \quad t > 0 \\
&u(0) = 1, \quad u'(0) = 2, \quad u''(0) = 3,
\end{aligned}
$$

find the approximate value of $u(h)$ based on the midpoint method with the step size $h = 0.2$.

2. (15 marks) Consider the ODE boundary value problem

$$
\begin{aligned}
&u'' + q(x)u = x, \quad 0 < x < 1, \\
&u(0) = u(1) = 0,
\end{aligned}
$$

where $q(x) = 1 + x$ for $x < 0.5$ and $q(x) = 1$ for $x > 0.5$. Describe a finite difference method for this problem using the grid size $h = 0.25$. Write down a linear system for u_1, u_2 and u_3, where $u_j \approx u(jh)$. You do not need to solve the linear system.

3. (20 marks) Consider the Crank-Nicolson method for the following initial and boundary value problem

$$
\begin{aligned}
&u_t = u_{xx}, \quad \text{for} \quad 0 < x < 1, \quad t > 0, \\
&u_x(0, t) = 0, \quad u(1, t) = 1, \quad t \geq 0, \\
&u(x, 0) = x, \quad 0 \leq x < 1.
\end{aligned}
$$

If we discretize x by $x_j = (j - 0.5)\Delta x$ (for $j = 1$, 2, 3, and $\Delta x = 1/3.5$) and discretize t by $t_k = k\Delta t$ (for $\Delta t = 8/49$), write down a linear system of equations for u_1^1, u_2^1 and u_3^1, where $u_j^k \approx u(x_j, t_k)$. You do not need to solve the system.

4. (15 marks) Following the derivations of the Lax-Wendroff method, derive a second order explicit finite difference method for the following differential equation

$$
u_t + u_x = u^2.
$$

5. (15 marks) For the first order partial differential equation $u_t = u_x$, the following numerical method is proposed:

$$
\frac{1}{\Delta t}\left[u_j^{k+1} - \frac{1}{2}(u_{j+1}^k + u_{j-1}^k)\right] = \frac{u_{j+1}^k - u_{j-1}^k}{2\Delta x}
$$

where $u_j^k \approx u(x_j, t_k)$, Δx and Δt are the grid size for x and step size for t, respectively. Show that the above numerical method is conditionally stable. Find the stability condition.

6. Consider the following linear system of equations

$$\begin{bmatrix} a & 1 & & & \\ 1 & a & 1 & & \\ & 1 & a & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & a \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{bmatrix}, \qquad (1)$$

where a is a given constant and the coefficient matrix is tridiagonal.

(a) (10 marks) If $\hat{u}_k$ and $\hat{f}_k$ (for $1 \le k \le n$) are the discrete sine transforms of u_j and f_j (for $1 \le j \le n$), show that

$$\hat{u}_k = \frac{\hat{f}_k}{a + 2\cos\frac{k\pi}{n+1}}.$$

(b) (5 marks) Find the eigenvalues and eigenvectors of the coefficient matrix in Eq. (1).

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