TO.................... FROM.................... DATE.................... NO. 1

MA3514 Exam Solutions
Semester B, 2006-2007
May 2007

1. We consider the local truncation error

$$T_{j+1} = y(t_j+h) - \left\{ \frac{4}{3} y(t_j) - \frac{1}{3} y(t_j-h) + \frac{2h}{3} f(t_{j+1}, y(t_{j+1})) \right\}$$

$$= y(t_j+h) - \frac{4}{3} y(t_j) + \frac{1}{3} y(t_j-h) - \frac{2h}{3} y'(t_j+h)$$

$$= y + hy' + \frac{h^2}{2} y'' + \frac{h^3}{6} y''' + \cdots$$

$$- \frac{4}{3} y + \frac{1}{3}\left[y - hy' + \frac{h^2}{2} y'' - \frac{h^3}{6} y''' + \cdots \right]$$

$$- \frac{2h}{3}\left[y' + hy'' + \frac{h^2}{2} y''' + \cdots \right]$$

here: $y = y(t_j)$, $y' = y'(t_j)$, $y'' = y''(t_j)$, ...

$$= -\frac{2}{9} h^3 y'''(t_j) + \cdots$$

Therefore, this is a second order method.

If we apply the method to $y' = 0$, we have

$$y_{j+1} - \frac{4}{3} y_j + \frac{1}{3} y_{j-1} = 0$$

If $y_j = A\lambda_1^j + B\lambda_2^j$, then λ_1, λ_2 satisfy

$$\lambda^2 - \frac{4}{3}\lambda + \frac{1}{3} = 0 \Rightarrow \lambda_1 = \frac{1}{3}, \ \lambda_2 = 1$$

$\Rightarrow y_j = A\left(\frac{1}{3}\right)^j + B$, it is always bounded, thus, the method is zero stable.

2. We will use Newton's method to solve

$$F(t) = 0$$

where $F(t) = u(\pi; t)$ and $u(x;t)$ satisfies the following initial value problem

$$\begin{cases} u''' + (1+\sin x)\, u'' = \dfrac{u'}{1+u^2}, & x > 0 \\ u\big|_{x=0} = 1 \\ u'\big|_{x=0} = 2 \\ u''\big|_{x=0} = t \end{cases}$$

Here, $u' = \dfrac{\partial u}{\partial x}$, $u'' = \dfrac{\partial^2 u}{\partial x^2}$, $u''' = \dfrac{\partial^3 u}{\partial x^3}$

Newton's method is:

$$t_{k+1} = t_k - \frac{F(t_k)}{F'(t_k)}.$$

To find $F'(t) = \dfrac{\partial u}{\partial t}(\pi; t)$, we define $v = \dfrac{\partial u}{\partial t}$, then v satisfies the I.V.P:

$$\begin{cases} v''' + (1+\sin x)\, v'' = \dfrac{v'}{1+u^2} - \dfrac{2u'uv}{(1+u^2)^2}, & x > 0 \\ v\big|_{x=0} = 0, \quad v'\big|_{x=0} = 0, \quad v''\big|_{x=0} = 1 \end{cases}$$

We can solve u and v together and find $F(t)$ and $F'(t)$.

3. If we let $u_j^k = \rho^k e^{ij\beta \cdot \Delta x}$, then

$$\rho - 1 = \frac{\Delta t}{2 \cdot \Delta x} \left[e^{i\beta \cdot \Delta x} - e^{-i\beta \cdot \Delta x} \right]$$

$$= \frac{\Delta t}{2 \cdot \Delta x} \, 2i \sin(\beta \cdot \Delta x)$$

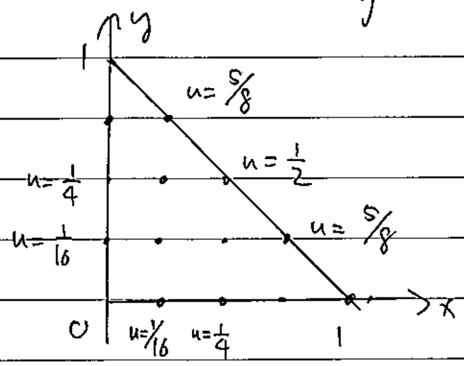
$$\Rightarrow |\rho|^2 = 1 + \left(\frac{\Delta t}{\Delta x}\right)^2 \sin^2(\beta \cdot \Delta x).$$

For any $\Delta t > 0$, we can also choose β such that $\sin(\beta \Delta x) \neq 0$ then $|\rho| > 1$.

Therefore the method is unstable for any $\Delta t > 0$, that is, the method is unconditionally unstable.

4. We have $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{ij} = 0$

where $u_{ij} \approx u(ih, jh)$, $h = \frac{1}{4}$.



from the boundary condition $u = x^2 + y^2$, we have

$U_{10} = \frac{1}{16}$, $U_{20} = \frac{1}{4}$

$U_{01} = \frac{1}{16}$, $U_{02} = \frac{1}{4}$

$U_{13} = \frac{5}{8}$, $U_{22} = \frac{1}{2}$, $U_{31} = \frac{5}{8}$.

we get:

$$\begin{bmatrix} -4 & 1 & 1 \\ 1 & -4 & 0 \\ 1 & 0 & -4 \end{bmatrix} \begin{bmatrix} U_{11} \\ U_{21} \\ U_{12} \end{bmatrix} = \begin{bmatrix} -\frac{1}{8} \\ -\frac{11}{8} \\ -\frac{11}{8} \end{bmatrix}.$$

5. From $V - s\frac{\partial^2 V}{\partial x^2} = W$, we get:

$$V_{ij} - \frac{s}{(\Delta x)^2}\left[V_{i-1,j} - 2V_{ij} + V_{i+1,j}\right] = W_{ij}$$

For the unit square and $\Delta x = \Delta y = \frac{1}{4}$, the above equation is valid for $1 \le i \le 3$ and $1 \le j \le 3$. Since $s = \frac{1}{16}$, $\Delta x = \frac{1}{4}$, we can simplify the above to:

$$-V_{i-1,j} + 3V_{ij} - V_{i+1,j} = W_{ij}$$

$$\Rightarrow \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix} = \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix} +$$

$$+\begin{bmatrix} V_{01} & V_{02} & V_{03} \\ 0 & 0 & 0 \\ V_{41} & V_{42} & V_{43} \end{bmatrix}$$

$$\Rightarrow \quad T = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}, \quad \tilde{W} = \begin{bmatrix} \frac{3}{16} & 0 & \frac{-5}{16} \\ \frac{7}{16} & \frac{1}{4} & \frac{-1}{16} \\ \frac{11}{16} & \frac{1}{2} & \frac{3}{16} \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

$$\Rightarrow \quad \tilde{W} = \begin{bmatrix} \frac{19}{16} & 1 & \frac{11}{16} \\ \frac{7}{16} & \frac{1}{4} & -\frac{1}{16} \\ \frac{15}{16} & 1 & \frac{15}{16} \end{bmatrix}.$$

6. We discretize the equation by:

$$\frac{1}{h^2}\left[U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{ij} \right] + \frac{1}{2h}\left(-U_{i-1,j} + U_{i+1,j} \right) = F_{ij}$$

Or: $$U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{ij} - \frac{h}{2} U_{i-1,j} + \frac{h}{2} U_{i+1,j} = h^2 F_{ij}$$

For $$U_{ij} = \sum_{k=1}^{n} \hat{u}_{ik} \sin \frac{jk\pi}{n+1}$$

TO.................. FROM.................. DATE.................. NO. 6

We get

$$\sum_{k=1}^{n}\left\{\left(1-\frac{h}{2}\right)\hat{u}_{i-1,k}+\left(1+\frac{h}{2}\right)\hat{u}_{i+1,k}-4\,\hat{u}_{ik}\right\}\sin\frac{jk\pi}{n+1}$$

$$+\sum_{k=1}^{n}\left\{\hat{u}_{i,k}\sin\frac{(j-1)k\pi}{n+1}+\hat{u}_{ik}\sin\frac{(j+1)k\pi}{n+1}\right\}$$

$$=h^2\sum_{k=1}^{n}\hat{F}_{ik}\sin\frac{jk\pi}{n+1}$$

We have: $\sin\frac{(j-1)k\pi}{n+1}+\sin\frac{(j+1)k\pi}{n+1}$

$$=2\sin\frac{jk\pi}{n+1}\cos\frac{k\pi}{n+1}$$

$$\Rightarrow:\ \sum_{k=1}^{n}\left\{\left(1-\frac{h}{2}\right)\hat{u}_{i-1,k}+\left(1+\frac{h}{2}\right)\hat{u}_{i+1,k}+\left(-4+2\cos\frac{k\pi}{n+1}\right)\hat{u}_{ik}-h^2\hat{F}_{ik}\right\}\sin\frac{jk\pi}{n+1}=0$$

$$\Rightarrow:\ \left(1-\frac{h}{2}\right)\hat{u}_{i-1,k}+\left(1+\frac{h}{2}\right)\hat{u}_{i+1,k}+\left(-4+2\cos\frac{k\pi}{n+1}\right)\hat{u}_{ik}=h^2\hat{F}_{ik}$$

$$\Rightarrow:\ a=-4+2\cos\frac{k\pi}{n+1},\quad b=1+\frac{h}{2},\quad c=1-\frac{h}{2}$$

$\beta_i=h^2\hat{F}_{ik}$, where

$$\hat{F}_{ik}=\frac{2}{n+1}\sum_{j=1}^{n}F_{ij}\sin\frac{jk\pi}{n+1}.$$