City University of Hong Kong

Course code & title:	MA3514 Numerical Methods for Differential Equations
Session:	Semester B, 2006-2007
Time allowed:	Three hours

This paper has **three** pages. (Including this page)

Instructions to candidates:

- Attempt **all** questions.
- Start each question on a new page.
- Show all working.

Materials, aids & instruments permitted to be used during examination:

- Non-programmable portable battery operated calculator.

1. (20 marks) Consider the following numerical method

$$y_{j+1} = \frac{4}{3}y_j - \frac{1}{3}y_{j-1} + \frac{2h}{3}f(t_{j+1}, y_{j+1})$$

for initial value problems of the ordinary differential equation $y' = f(t, y)$. Find the local truncation error and the order of the method. Is this method zero-stable?

2. (15 marks) Consider the boundary value problem

$$\begin{aligned} &y''' + (1 + \sin x)y'' = \frac{y'}{1+y^2}, \quad 0 < x < \pi, \\ &y(0) = 1,\ y'(0) = 2,\ y(\pi) = 0. \end{aligned}$$

Describe how shooting method (together with Newton's method for solving nonlinear equations) can be used to solve this boundary value problem.

3. (15 marks) For the differential equation $u_t = u_x$, a possible numerical method is

$$\frac{u_j^{k+1} - u_j^k}{\Delta t} = \frac{u_{j+1}^k - u_{j-1}^k}{2\Delta x},$$

where $u_j^k \approx u(x_j, t_k)$ for $t_k = t_0 + k\Delta t$ and $x_j = x_0 + j\Delta x$. Analyze the stability of this method by considering special solutions given by

$$u_j^k = \rho^k e^{\mathbf{i}j\beta\Delta x}.$$

4. (20 marks) Consider the Laplace equation

$$u_{xx} + u_{yy} = 0$$

on the triangle bounded by the x-axis, the y-axis and the line $x + y = 1$. The boundary condition is $u(x, y) = x^2 + y^2$ if (x, y) is on the boundary of the triangle. If a second order finite difference method is used to discretize the equation with $\Delta x = \Delta y = h = 1/4$, find the matrix A and vector b, such that

$$A \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \end{bmatrix} = b,$$

where $u_{ij} \approx u(ih, jh)$. You do not need to solve the system.

5. (15 marks) As a sub-step of the ADI method for $u_t = u_{xx} + u_{yy}$, we need to solve $v = v(x, y)$ from

$$v - s\partial_x^2 v = w(x, y).$$

Let us assume that this problem is to be solved on the unit square: $0 < x < 1$ and $0 < y < 1$, the function w is given by $w = x - y^2$ and v satisfies the following boundary conditions: $v = 1$ for $x = 0$ and $v = y$ for $x = 1$. If we discretize x and y by $x_i = i/4$ and $y_j = j/4$ and approximate the equation by a second order finite difference method, we obtain

$$TV = \tilde{W},$$

where V is the 3×3 matrix of $v_{ij} \approx v(x_i, y_j)$. Write down the matrices T and $\tilde{W}$ for $s = 1/16$.

6. (15 marks) Let $F(x, y)$ be a given function, we consider the equation

$$u_{xx} + u_{yy} + u_x = F(x, y)$$

on the unit square

$$\Omega = \{\, (x, y) \,|\, 0 < x < 1,\ 0 < y < 1 \,\}$$

with the boundary condition $u|_{\partial\Omega} = 0$. The equation is approximated by a second order finite difference method based on the discretization

$$x_i = ih, \quad y_j = jh, \quad h = \frac{1}{n+1}$$

for some integer n. Let $u_{ij} \approx u(x_i, y_j)$ be the numerical approximation and define $\hat{u}_{ik}$ by the discrete sine transform

$$u_{ij} = \sum_{k=1}^{n} \hat{u}_{ik} \sin \frac{jk\pi}{n+1}.$$

Show that $\hat{u}_{ik}$ satisfies

$$\begin{bmatrix} a & b & & \\ c & a & \ddots & \\ & \ddots & \ddots & b \\ & & c & a \end{bmatrix} \begin{bmatrix} \hat{u}_{1k} \\ \hat{u}_{2k} \\ \vdots \\ \hat{u}_{nk} \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}.$$

Write down the formulas for a, b, c and β_i.

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