MA3514 Exam Solutions
Semester B, 2005-2006

1. Let $y' = u \Rightarrow \frac{d}{dt}\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} u \\ -y^2+u+t \end{bmatrix}$

$$= f\left(t, \begin{bmatrix} y \\ u \end{bmatrix}\right)$$

$$\begin{bmatrix} y_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$$\vec{k}_1 = f\left(t_0, \begin{bmatrix} y_0 \\ u_0 \end{bmatrix}\right) = f\left(0, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1^2+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\vec{k}_2 = f\left(t_0+\frac{h}{2}, \begin{bmatrix} y_0 \\ u_0 \end{bmatrix} + \frac{h}{2}\vec{k}_1\right)$$

$$= f\left(0.05, \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0.05\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

$$= f\left(0.05, \begin{bmatrix} 1.1 \\ 2.05 \end{bmatrix}\right) = \begin{bmatrix} 2.05 \\ -1.1^2+2.05+0.05 \end{bmatrix}$$

$$= \begin{bmatrix} 2.05 \\ 0.89 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ u_0 \end{bmatrix} + h\,k_2 = \begin{bmatrix} 1.205 \\ 2.089 \end{bmatrix}$$

$$\Rightarrow \quad y(0.1) \approx y_1 = 1.205$$

2. We have Taylor series:

$$y(h) = y(0) + h\,y'(0^+) + \frac{h^2}{2} y''(0^+) + \cdots$$
$$y(-h) = y(0) - h\,y'(0^-) + \frac{h^2}{2} y''(0^-) + \cdots$$

$$\rho\, y(h) + y(-h) = (\rho+1) y(0) + \frac{h^2}{2}\left[\rho\, y''(0^+) + y''(0^-)\right] + \cdots$$

Next, we use the differential equation at $x = 0^+$ and $x = 0^-$,

thus: $$y''(0^-) + q(0^-)\, y(0) = r(0)$$
$$y''(0^+) + q(0^+)\, y(0) = r(0)$$

$$\Rightarrow \quad \rho\, y(h) + y(-h) = (\rho+1)\, y(0) + \frac{h^2}{2}\Big[\rho\, r(0) - \rho\, q(0^-) y(0) + r(0) - q(0^+) y(0)\Big] + \cdots$$

$$\therefore \quad \rho\, y(h) + y(-h) - (\rho+1)\, y(0) + \frac{h^2}{2} \rho\, q(0^-)\, y(0) + \frac{h^2}{2} q(0^+)\, y(0) = \frac{h^2}{2}(\rho+1)\, r(0) + \cdots$$

$$\Rightarrow: \quad \frac{1}{h^2}\left[\frac{2}{\rho+1} y(-h) - 2 y(0) + \frac{2\rho}{\rho+1} y(h)\right] + \left[\frac{q(0^+)}{\rho+1} + \frac{\rho}{\rho+1} q(0^-)\right] y(0) = r(0)$$

thus, $a y_{-1} + b y_0 + c y_1 = d$

where $a = \frac{2}{(\rho+1)h^2}$, $b = \frac{-2}{h^2} + \frac{q(0^+) + \rho\, q(0^-)}{\rho+1}$

$c = \frac{2\rho}{(\rho+1)h^2}$, $d = f(0)$

3. The Crank-Nicolson method is:

$$\frac{1}{\Delta t}\left[u_j^1 - u_j^0\right] = \frac{(1+x_j)}{2(\Delta x)^2}\left[u_{j-1}^1 - 2u_j^1 + u_{j+1}^1 + u_{j-1}^0 - 2u_j^0 + u_{j+1}^0\right]$$

Boundary Conditions: $u_0^k = u_4^k = 0$, $\Delta x = \frac{1}{4}$

Initial Conditions: $u_1^0 = \frac{1}{2}$ $u_2^0 = 1$, $u_3^0 = \frac{1}{2}$

For $j=1$:

$$u_1^1 - \frac{1}{2} = \frac{5}{2}\left[-2u_1^1 + u_2^1 + 0\right]$$

$$\Rightarrow 6u_1^1 - \frac{5}{2}u_2^1 = \frac{1}{2}$$

For $j=2$:

$$u_2^1 - 1 = 3\left[u_1^1 - 2u_2^1 + u_3^1 + (-1)\right]$$

$$\Rightarrow -3u_1^1 + 7u_2^1 - 3u_3^1 = -2$$

For $j=3$: $u_3^1 - \frac{1}{2} = \frac{7}{2}\left[u_2^1 - 2u_3^1 + 0\right]$

$$\Rightarrow: \ -\frac{7}{2} u_2^1 + 8 u_3^1 = \frac{1}{2}$$

$$\Rightarrow \begin{bmatrix} 6 & -\frac{5}{2} & 0 \\ -3 & 7 & -3 \\ 0 & -\frac{7}{2} & 8 \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -2 \\ \frac{1}{2} \end{bmatrix}$$

4. For $u_t + a(x) u_x = 0$, we have

$$u(x, t+\Delta t) = u(x,t) + \Delta t \cdot u_t(x,t) + \frac{(\Delta t)^2}{2} u_{tt}(x,t) + \cdots$$

$$\approx u - \Delta t \cdot a(x) u_x + \frac{(\Delta t)^2}{2} a(x) \left(a(x) u_x\right)_x$$

$\Rightarrow$: Lax-Wendroff method is

$$u_j^1 = u_j^0 - \frac{\Delta t \, a(x_j)}{2 \Delta x} \cdot \left[u_{j+1}^0 - u_{j-1}^0\right]$$

$$+ \frac{(\Delta t)^2}{2 (\Delta x)^2} a(x_j) \left[a(x_{j+\frac{1}{2}}) (u_{j+1}^0 - u_j^0) - a(x_{j-\frac{1}{2}}) (u_j^0 - u_{j-1}^0) \right]$$

Here: $u_j^0 = f(x_j)$, $\Rightarrow$ $u_1^0 = 0.2$, $u_0^0 = 0$, $u_{-1}^0 = 0$

$a(x) = 1 + x$ $\Rightarrow$ $a(x_0) = 1$, $a(x_{0+\frac{1}{2}}) = 1.1$

$a(x_{0-\frac{1}{2}}) = 0.9$

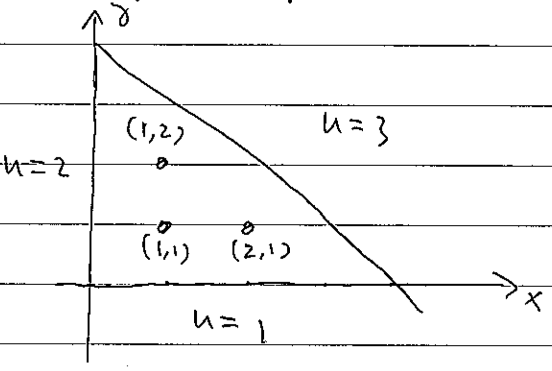
$$\Rightarrow \ u_0^1 = 0 - \frac{1}{4} \times 0.2 + \frac{1}{8} \left[1.1 \times 0.2 - 0.9 \times 0 \right]$$

$$= -0.0225$$

5. At (x_i, y_j), $x_i = ih$, $y_j = ih$

We have:

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{ij} + h^2 \cdot x_i \, u_{ij} = h^2 y_j$$



thus:

at (x_1, y_1):

$$1 + 2 - 4u_{11} + u_{21} + u_{12} + \frac{1}{64} u_{11} = \frac{1}{64}$$

at (x_2, y_1):

$$1 + 3 + 3 - 4u_{21} + u_{11} + \frac{1}{32} u_{21} = \frac{1}{64}$$

at (x_1, y_2):

$$2 + 3 + 3 - 4u_{12} + u_{11} + \frac{1}{64} u_{12} = \frac{1}{32}$$

$$\Rightarrow: \begin{bmatrix} -\frac{255}{64} & 1 & 1 \\ 1 & -\frac{127}{32} & 0 \\ 1 & 0 & -\frac{255}{64} \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \end{bmatrix} = \begin{bmatrix} -\frac{191}{64} \\ -\frac{447}{64} \\ -\frac{255}{32} \end{bmatrix}$$

6. (a) $$\frac{1}{\Delta t}\left[E_j^{k+\frac{1}{2}} - E_j^{k-\frac{1}{2}}\right] = -\frac{1}{\varepsilon}\cdot\frac{1}{\Delta z}\left[H_{j+\frac{1}{2}}^{k} - H_{j-\frac{1}{2}}^{k}\right]$$

$$\frac{1}{\Delta t}\left[H_{j+\frac{1}{2}}^{k+1} - H_{j+\frac{1}{2}}^{k}\right] = -\frac{1}{\mu}\cdot\frac{1}{\Delta z}\left[E_{j+1}^{k+\frac{1}{2}} - E_j^{k+\frac{1}{2}}\right]$$

(b) Insert the special soltions into the above numerical scheme, we get:

$$\frac{E_*}{\Delta t}\left(\sqrt{\rho} - \frac{1}{\sqrt{\rho}}\right) = -\frac{1}{\varepsilon}\cdot\frac{H_*}{\Delta z}\cdot\left[e^{i\frac{\beta\Delta z}{2}} - e^{-\frac{i\beta\cdot\Delta z}{2}}\right]$$

$$\frac{H_*}{\Delta t}\left[\sqrt{\rho} - \frac{1}{\sqrt{\rho}}\right] = -\frac{1}{\mu}\cdot\frac{E_*}{\Delta z}\left[e^{i\frac{\beta\Delta z}{2}} - e^{-\frac{i\beta\Delta z}{2}}\right]$$

$$\Rightarrow \frac{H_*}{\varepsilon E_*} = \frac{1}{\mu}\frac{E_*}{H_*} = \frac{1}{\sqrt{\varepsilon\mu}} = c$$

$$\Rightarrow \sqrt{\rho} - \frac{1}{\sqrt{\rho}} = -\frac{c\cdot\Delta t}{\Delta z}\left[e^{i\frac{\beta\cdot\Delta z}{2}} - e^{-\frac{i\beta\Delta z}{2}}\right]$$

$$= -\frac{c\cdot\Delta t}{\Delta z}\,2i\sin\left(\frac{\beta\cdot\Delta z}{2}\right) = -2\tau i$$

where $\tau = c\cdot\frac{\Delta t}{\Delta z}\sin\left(\frac{\beta\cdot\Delta z}{2}\right)$

$$\Rightarrow \sqrt{\rho} - \frac{1}{\sqrt{\rho}} = -2\tau i \Rightarrow \rho + 2i\tau\sqrt{\rho} - 1 = 0$$

$$\Rightarrow \sqrt{\rho} = -i\tau \pm \sqrt{-\tau^2+1}$$

If: $c\cdot\frac{\Delta t}{\Delta z} \leq 1$ then $|\tau| \leq 1$

$$\Rightarrow |\sqrt{\rho}| = 1 \Rightarrow |\rho| = 1$$

$\Rightarrow$ stable

If $c \cdot \frac{\Delta t}{\Delta z} > 1$ then, we can choose β such that $|\tau| > 1$, then

one of the two solutions of $\sqrt{\rho}$ satisfies $|\sqrt{\rho}| > 1$ or $|\rho| > 1$

$\Rightarrow$ unstable.

In conclusion, the numerical method

is stable if $c \cdot \frac{\Delta t}{\Delta z} \le 1$

is unstable if $c \cdot \frac{\Delta t}{\Delta z} > 1$

where $c = \frac{1}{\sqrt{\varepsilon \mu}}$.