MA 3514, Exam Solutions
Semester B, 2004-2005.

Q1: $\frac{d\vec{y}}{dt} = \vec{f}(t, \vec{y})$, $\quad \vec{y} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$, $\quad \vec{f}(t, \vec{y}) = \begin{bmatrix} y' \\ y'' \\ 2y + (y'')^2 + t \end{bmatrix}$

$$\vec{y}_0 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \vec{k}_1 = \vec{f}(0, \vec{y}_0) = \begin{bmatrix} 0 \\ -2 \\ 6 \end{bmatrix}, \quad t_0 = 0,\ h = 0.2$$

$$\vec{k}_1 = \vec{f}\left(t_0 + \frac{h}{2},\ \vec{y}_0 + \frac{h}{2}\vec{k}_1\right) = \vec{f}\left(0.1,\ \begin{bmatrix} 1 \\ -0.2 \\ -1.4 \end{bmatrix}\right) = \begin{bmatrix} -0.2 \\ -1.4 \\ 4.06 \end{bmatrix}$$

$$\vec{y}_1 = \vec{y}_0 + h\,\vec{k}_2 = \begin{bmatrix} 0.96 \\ -0.28 \\ -1.188 \end{bmatrix}.$$

Q2: We define an I.V.P for $U = U(x; \lambda)$

$$\begin{cases} \frac{\partial^2 U}{\partial x^2} + [\lambda p(x) + \lambda^2 q(x)]\, U = 0, & x > 0 \\ U(0; \lambda) = 0 \\ \frac{\partial U}{\partial x}(0; \lambda) = 1 \end{cases}$$

then: $\quad \phi(\lambda) = U(1; \lambda)$

For $\phi'(\lambda)$, let $\quad V = \frac{\partial U}{\partial \lambda}$

$$\Rightarrow \begin{cases} \frac{\partial^2 V}{\partial x^2} + [p(x) + 2\lambda q(x)]\, U + [\lambda p(x) + \lambda^2 q(x)]\, V = 0 \\ V(0; \lambda) = 0 \\ \frac{\partial V}{\partial x}(0; \lambda) = 0 \end{cases}$$

then: $\quad \phi'(\lambda) = V(1; \lambda)$

Q3: For $x_j = jh$, we have

$$\phi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h}, & x_{j-1} < x < x_j \\ \dfrac{x_{j+1} - x}{h}, & x_j < x < x_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

$$\phi_j'(x) = \begin{cases} \dfrac{1}{h}, & x_{j-1} < x < x_j \\ -1/h, & x_j < x < x_{j+1} \\ 0. & \text{otherwise, except at } x_k, \ k = \text{integers} \end{cases}$$

$$\Rightarrow: \int \phi_j \phi_{j-1}\, dx = \int_{x_{j-1}}^{x_j} \frac{x - x_{j-1}}{h} \cdot \frac{x_j - x}{h}\, dx = h \int_0^1 t(1-t)\, dt$$

$$= h/6. \qquad \text{, where } t = \frac{x - x_{j-1}}{h}$$

Similarly: $\int \phi_j \phi_{j+1}\, dx = \dfrac{h}{6}$

$$\int \phi_j^2\, dx = \int_{x_{j-1}}^{x_j} \left(\frac{x - x_{j-1}}{h}\right)^2 dx + \int_{x_j}^{x_{j+1}} \left(\frac{x_{j+1} - x}{h}\right)^2 dx$$

$$= \frac{2}{3} h$$

$$\int \phi_j' \phi_{j-1}'\, dx = \int_{x_{j-1}}^{x_j} \frac{1}{h} \cdot \left(\frac{-1}{h}\right) dx = -\frac{1}{h}$$

$$\int \phi_j' \phi_{j+1}'\, dx = -\frac{1}{h}$$

$$\int \phi_j' \phi_j'\, dx = \int_{x_{j-1}}^{x_j} \left(\frac{1}{h}\right)^2 dx + \int_{x_j}^{x_{j+1}} \left(\frac{-1}{h}\right)^2 dx = \frac{2}{h}.$$

Q4: Let $u_j^k \approx u(x_j, t_k)$

then: $\frac{1}{\Delta t}\left[u_j^1 - u_j^0\right] = (1+t_{1/2})\frac{1}{2(\Delta x)^2}\left[u_{j-1}^1 - 2u_j^1 + u_{j+1}^1 + u_{j-1}^0 - 2u_j^0 + u_{j+1}^0\right]$

let $\gamma = \frac{(\Delta t)(1+t_{1/2})}{2(\Delta x)^2}$, then:

$$-\gamma u_{j-1}^1 + (1+2\gamma)u_j^1 - \gamma u_{j+1}^1 = \gamma u_{j-1}^0 + (1-2\gamma)u_j^0 + \gamma u_{j+1}^0.$$

Now: $\Delta x = \frac{1}{3}$, $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$, $x_3 = 1$

the above equation is needed for $j=1$, $j=2$.

$$\begin{bmatrix} 1+2\gamma, & -\gamma \\ -\gamma, & 1+2\gamma \end{bmatrix}\begin{bmatrix} u_1^1 \\ u_2^1 \end{bmatrix} = \begin{bmatrix} \gamma u_0^0 + (1-2\gamma)u_1^0 + \gamma u_2^0 \\ \gamma u_1^0 + (1-2\gamma)u_2^0 + \gamma u_3^0 \end{bmatrix} + \begin{bmatrix} \gamma u_0^1 \\ \gamma u_3^1 \end{bmatrix}$$

From the boundary conditions: $u_0^0 = 0$, $u_0^1 = 0$
$u_3^0 = 2$, $u_3^1 = 2$

From the initial conditions: $u_1^0 = 1$, $u_2^0 = 1$

$$\Rightarrow: \begin{pmatrix} 1+2\gamma & -\gamma \\ -\gamma & 1+2\gamma \end{pmatrix}\begin{bmatrix} u_1^1 \\ u_2^1 \end{bmatrix} = \begin{pmatrix} 1-\gamma \\ 1+3\gamma \end{pmatrix}$$

$$\gamma = \frac{\Delta t \cdot (1+t_{1/2})}{2\cdot(\Delta x)^2} = \frac{\frac{1}{3}\cdot\left(1+\frac{1}{6}\right)}{2\cdot\left(\frac{1}{3}\right)^2} = \frac{7}{4}$$

$$\Rightarrow \quad u_1^1 = 1.44$$
$$u_2^1 = 1.56$$

Q5: If we insert $u_j^k = \rho^k e^{i\beta x_j}$ into the method, we get

$$\rho - 2 + \frac{1}{\rho} = s^2 \left[e^{i\beta\cdot\Delta x} - \left(\rho + \frac{1}{\rho}\right) + e^{-i\beta\cdot\Delta x} \right].$$

where $s = \dfrac{c\cdot\Delta t}{\Delta x} > 0$

$$\Rightarrow \quad \rho - 2 + \frac{1}{\rho} = s^2 \left[2\cos(\beta\cdot\Delta x) - \left(\rho + \frac{1}{\rho}\right) \right]$$

$$\Rightarrow \quad (1+s^2)\left[\rho + \frac{1}{\rho}\right] = 2 + 2s^2\cos(\beta\cdot\Delta x)$$

$$\rho + \frac{1}{\rho} = 2\cdot\frac{1+s^2\cos(\beta\cdot\Delta x)}{1+s^2} = 2\gamma$$

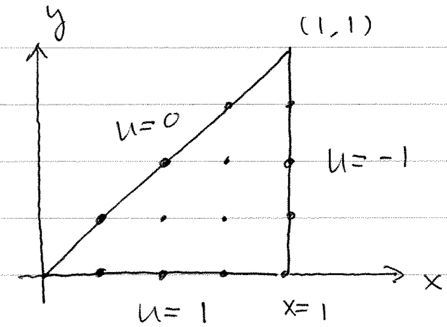
since $|\cos(\beta\cdot\Delta x)| \le 1 \Rightarrow |\gamma| \le 1$

$$\Rightarrow \quad \rho + \frac{1}{\rho} = 2\gamma \Rightarrow \rho^2 - 2\gamma\rho + 1 = 0 \Rightarrow$$

$$\rho = \gamma \pm \sqrt{1-\gamma^2}\, i$$

$$\Rightarrow \quad |\rho|^2 = \gamma^2 + (1-\gamma^2) = 1 \Rightarrow |\rho| = 1$$

Q6. We have



the discretization of $u_{xx} + u_{yy} = 0$ gives:

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{ij} = 0$$

from the boundary conditions:

$$u_{10} = u_{20} = u_{30} = 1$$

$$u_{41} = u_{42} = u_{43} = -1$$

$$u_{11} = u_{22} = u_{33} = 0$$

at (x_2, y_1): $\quad 0 + 1 + u_{31} + 0 - 4u_{21} = 0$

at (x_3, y_1): $\quad u_{21} + 1 + (-1) + u_{32} - 4u_{31} = 0$

at (x_3, y_2): $\quad 0 + u_{31} + (-1) + 0 - 4u_{32} = 0$

$$\Rightarrow: \begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{31} \\ u_{32} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow: \quad u_{21} = \frac{1}{4}, \quad u_{31} = 0, \quad u_{32} = -\frac{1}{4}.$$