# City University of Hong Kong

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| Course code & title: | MA3514 Numerical Methods for Differential Equations |
| Session: | Semester B, 2004-2005 |
| Date: | 6 May 2005 |
| Time: | 18:30–21:30 |
| Time allowed: | Three hours |

This paper has **three** pages. (Including this page)

Instructions to candidates:

- Attempt **all** questions.
- Start each question on a new page.
- Show all working.

Materials, aids & instruments permitted to be used during examination:

- Non-programmable portable battery operated calculator.

1. (15 marks) Consider the following initial value problem

$$\begin{aligned} y''' &= 2y + (y'')^2 + t, \quad t > 0, \\ y(0) &= 1, \\ y'(0) &= 0, \\ y''(0) &= -2. \end{aligned}$$

   Use the midpoint method with step size $h = 0.2$, find an approximate solution of $y(h)$.

2. (15 marks) Consider the quadratic eigenvalue problem

$$\begin{aligned} u'' + [\lambda p(x) + \lambda^2 q(x)]u &= 0, \\ u(0) = u(1) &= 0, \end{aligned}$$

   where $p$ and $q$ are given functions, $\lambda$ is the unknown eigenvalue and $u$ is the unknown eigenfunction. In a shooting method, we solve a nonlinear equation $\phi(\lambda) = 0$. Give a precise definition of $\phi$. How to calculate $\phi'(\lambda)$?

3. (15 marks) Let $\phi_j(x)$ be the continuous and piecewise linear function (as used in the finite element method) satisfying

$$\phi_j(x_j) = 1, \quad \text{and} \quad \phi_j(x_k) = 0, \quad j \neq k.$$

   If the discrete nodes are given by $x_j = jh$ for a constant $h$, find

$$\int \phi_j'(x)\phi_k'(x)dx \quad \text{and} \quad \int \phi_j(x)\phi_k(x)dx$$

   for $k = j-1$, $j$ and $j+1$.

4. (15 marks) Consider the parabolic equation

$$u_t = (1+t)u_{xx}, \quad 0 < x < 1,$$

   subject to the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 2,$$

   and the initial condition

$$u(x,0) = 1.$$

   Use Crank-Nicolson's method with $\Delta x = 1/3$ and $\Delta t = 1/3$, find the approximate values of $u(x_j, t_k)$ for $j = 1, 2$ and $k = 1$, where $x_j = j\Delta x$ and $t_k = k\Delta t$.

5. (20 marks) Consider the wave equation $u_{tt} = c^2 u_{xx}$ (where $c > 0$ is a constant) and the numerical method:

$$\frac{1}{(\Delta t)^2}\left[u_j^{k+1} - 2u_j^k + u_j^{k-1}\right] = \frac{c^2}{(\Delta x)^2}\left[u_{j+1}^k - (u_j^{k+1} + u_j^{k-1}) + u_{j-1}^k\right],$$

where $u_j^k \approx u(x_j, t_k)$ and $x_j = j\Delta x$, $t_k = k\Delta t$. For a special solution

$$u_j^k = \rho^k e^{i\beta x_j}, \quad i = \sqrt{-1},$$

show that $|\rho| = 1$ for any real $\beta$. Therefore, the numerical method is unconditionally stable.

6. (20 marks) Consider the Laplace equation

$$u_{xx} + u_{yy} = 0$$

inside the triangle connecting the three points: $(0, 0)$, $(1, 0)$ and $(1, 1)$. Let the boundary conditions be

$$\begin{aligned} u = 1, &\qquad \text{for} \quad y = 0, \\ u = -1, &\qquad \text{for} \quad x = 1, \\ u = 0, &\qquad \text{for} \quad y = x. \end{aligned}$$

Use a second order finite difference method with $\Delta x = \Delta y = 1/4$, find the approximations for $u(0.5, 0.25)$, $u(0.75, 0.25)$ and $u(0.75, 0.5)$.

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