

MA3514 — Assignment No. 6

1. Consider the Poisson equation

$$u_{xx} + u_{yy} = x$$

on the triangle Ω (bounded by the x -axis, the y -axis and the line $x + y = 1$), subject to the boundary condition $u|_{\partial\Omega} = g$, where $g = 1$ on the x -axis, $g = 0$ on the y -axis and $g = 2$ on the line $x + y = 1$. Let $h = 1/4$ and $u_{ij} \approx u(ih, jh)$ be the numerical solution obtained from a second order finite difference method. Define a vector \vec{u} for the unknowns, find a matrix A and a vector \vec{b} , such that $A\vec{u} = \vec{b}$. You do not need to solve the linear system.

2. Let $\rho = \rho(y)$ and $F = F(x, y)$ be given functions, consider the equation

$$u_{xx} + u_{yy} + \rho(y)u = F(x, y)$$

on the unit square

$$\Omega = \{ (x, y) \mid 0 < x < 1, 0 < y < 1 \}$$

with the boundary condition $u|_{\partial\Omega} = 0$, where $\partial\Omega$ is the boundary of Ω . If the equation is approximated by a second order finite difference method, describe a fast solver based on Discrete Sine Transform.

3. For the ODE boundary value problem

$$u''(x) = f(x), \quad 0 < x < 1, \quad u(0) = \alpha, \quad u(1) = \beta,$$

where f is a given function, α and β are given constants, the second order finite difference method gives

$$A\vec{U} = \vec{b}, \quad \text{where} \quad A = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & & 1 & -2 \\ & & & & 1 & -2 \end{bmatrix}.$$

Show that the Gauss-Seidel iterative method converges for the system above.

4. Based on the following definitions and properties of the conjugate gradient method

$$\begin{aligned} \mathcal{K}_n &= \langle b, Ab, A^2b, \dots, A^{n-1}b \rangle, & \phi(x_n) &= \min_{x \in \mathcal{K}_n} \phi(x) \\ r_n \perp \mathcal{K}_n &= \langle r_0, r_1, \dots, r_{n-1} \rangle = \langle x_1, x_2, \dots, x_n \rangle = \langle p_0, p_1, \dots, p_{n-1} \rangle, \end{aligned}$$

prove the following formulae for α_n and β_n :

$$\begin{aligned} \alpha_n &= \frac{p_{n-1}^T b}{p_{n-1}^T A p_{n-1}} = \frac{r_{n-1}^T p_{n-1}}{p_{n-1}^T A p_{n-1}} = \frac{r_{n-1}^T r_{n-1}}{p_{n-1}^T A p_{n-1}} \\ \beta_n &= -\frac{r_n^T A p_{n-1}}{p_{n-1}^T A p_{n-1}} = \frac{r_n^T r_n}{r_{n-1}^T r_{n-1}}. \end{aligned}$$