

MA3514 — Assignment No. 6

1. Consider the equation

$$u_{xx} + u_{yy} + xu = 0$$

on the triangular region $\Omega = \{(x, y) \mid x > 0, y > 0, x + y < 1\}$, subject to the boundary condition $u|_{\partial\Omega} = g$, where $g = 1$ on the x -axis, $g = 2$ on the y -axis and $g = 3$ on the line $x + y = 1$. Let $h = 1/4$ and $u_{ij} \approx u(ih, jh)$ be the numerical solution obtained from a second order finite difference method. Define a vector \vec{u} for the unknowns and find a matrix A and a vector \vec{b} such that $A\vec{u} = \vec{b}$.

2. Let $\rho(x)$ and $F(x, y)$ be given functions, consider the equation

$$u_{xx} + u_{yy} + \rho(x)u = F(x, y)$$

on the unit square

$$\Omega = \{(x, y) \mid 0 < x < 1, 0 < y < 1\}$$

with the boundary condition $u|_{\partial\Omega} = 0$. The equation is approximated by a second order finite difference method based on the discretization

$$x_i = ih, \quad y_j = jh, \quad h = \frac{1}{n+1}$$

for some integer n . Let $u_{ij} \approx u(x_i, y_j)$ be the numerical approximation and define \hat{u}_{ik} by the discrete sine transform

$$u_{ij} = \sum_{k=1}^n \hat{u}_{ik} \sin \frac{jk\pi}{n+1}.$$

Show that \hat{u}_{ik} satisfies

$$\begin{bmatrix} \alpha_1 & 1 & & \\ 1 & \alpha_2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & \alpha_n \end{bmatrix} \begin{bmatrix} \hat{u}_{1k} \\ \hat{u}_{2k} \\ \vdots \\ \hat{u}_{nk} \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}.$$

Write down the formulas giving α_i and β_i .

3. Let $\Omega = \{(x, y) \mid 0 < x < 1, 0 < y < 1\}$ be the unit square. Consider the following equation

$$u_{xx} + u_{yy} - u = x - y, \quad \text{for } (x, y) \in \Omega$$

with the boundary condition

$$u = x^2 + y^2, \quad \text{for } (x, y) \in \partial\Omega.$$

Let $u_{ij} \approx u(i/3, j/3)$, $0 \leq i, j \leq 3$, be the numerical solution based on a second order finite difference method. Find a matrix A and a vector \vec{b} , such that $A\vec{u} = \vec{b}$, where

$$\vec{u} = [u_{11}, u_{21}, u_{12}, u_{22}]^T.$$

You do not need to solve \vec{u} .

4. Consider the Poisson equation

$$u_{xx} + u_{yy} = x$$

on the triangle Ω (bounded by the x -axis, the y -axis and the line $x+y = 1$), subject to the boundary condition $u|_{\partial\Omega} = g$, where $g = 1$ on the x -axis, $g = 0$ on the y -axis and $g = 2$ on the line $x + y = 1$. Let $h = 1/4$ and $u_{ij} \approx u(ih, jh)$ be the numerical solution obtained from a second order finite difference method. Define a vector \vec{u} for the unknowns, find a matrix A and a vector \vec{b} , such that $A\vec{u} = \vec{b}$.