

## MA3514 — Assignment No. 4

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1. For the heat equation  $u_t = au_{xx}$  where  $a$  is a positive constant, show that the Dufort-Frankel method

$$\frac{1}{2\Delta t}(u_j^{k+1} - u_j^{k-1}) = \frac{a}{(\Delta x)^2}(u_{j+1}^k - u_j^{k+1} - u_j^{k-1} + u_{j-1}^k).$$

is unconditionally stable.

2. Consider the equation  $u_t = u_{xx} + bu_x$ , where  $b \neq 0$  is a real constant. Analyze the stability of the following numerical method (for the above equation):

$$\frac{u_j^{k+1} - u_j^k}{\Delta t} = \frac{1}{(\Delta x)^2}(u_{j+1}^k - 2u_j^k + u_{j-1}^k) + \frac{b}{2\Delta x}(u_{j+1}^k - u_{j-1}^k).$$

3. Consider the following initial and boundary value problem

$$\begin{aligned} u_t &= u_{xx} \quad \text{for } 0 < x < 1, \quad t > 0 \\ u(1, t) &= u(0, t), \quad u_x(1, t) = u_x(0, t), \quad t > 0, \\ u(x, 0) &= f(x), \quad 0 \leq x \leq 1. \end{aligned}$$

The periodic boundary conditions above can also be replaced by  $u(x+1, t) = u(x, t)$  for any  $x$ . If the Crank-Nicolson method is applied to the above problem with  $x_j = j/4$  and  $t_k = k/9$ , find the matrix  $A$ , such that  $\mathbf{v}_{k+1} = A\mathbf{v}_k$ , where  $\mathbf{v}_k = [u_1^k, u_2^k, u_3^k, u_4^k]^T$  and  $u_j^k \approx u(x_j, t_k)$ .

4. For the 2D heat equation on the unit square  $\Omega = \{(x, y) \mid 0 < x < 1, 0 < y < 1\}$ ,

$$u_t = u_{xx} + u_{yy}, \quad (x, y) \in \Omega, \quad t > 0,$$

with the boundary condition  $u(x, y, t) = 0$  for  $(x, y) \in \partial\Omega$  and  $t \geq 0$  and the initial condition

$$u(x, y, 0) = \begin{cases} 1, & \text{if } (x - 0.5)^2 + (y - 0.5)^2/4 < 1/9, \\ 0, & \text{otherwise.} \end{cases}$$

Write a MATLAB program for the ADI method based on the matrix notations. Choose  $\Delta x = \Delta y = 1/101$  and  $\Delta t = 0.0005$ , find the solutions at  $t = 0.001$  and  $t = 0.01$ . Submit the MATLAB program and a plot of the solutions (by MATLAB command `imagesc`).