

MA3514 — Assignment No. 2

1. The Korteweg-de Vries (KdV) equation (1895)

$$u_t + uu_x + \delta^2 u_{xxx} = 0,$$

where δ is a constant, models water waves in a shallow canal. The KdV equation has soliton solutions which were first observed by J. Scott Russell on the Edinburgh-Glasgow canal in 1834. Many mathematical properties of the KdV equation were found after the initial numerical study of Zabusky and Kruskal in 1965. They solved the KdV equation for $\delta = 0.022$ and $0 < x \leq 2$ assuming periodic boundary condition: $u(x+2, t) = u(x, t)$ for all x , with the initial condition: $u(x, 0) = \cos(\pi x)$, and for $0 < t \leq 3.6/\pi$. We will repeat this calculation with our own method. Discretizing x by $x_k = 2k/n$ for $n = 200$ and $1 \leq k \leq n$, the KdV equation is approximated by the following system of ODEs:

$$\frac{du_k}{dt} + u_k \frac{u_{k+1} - u_{k-1}}{2d} + \frac{\delta^2}{2d^3} (u_{k+2} - 2u_{k+1} + 2u_{k-1} - u_{k-2}) = 0,$$

for $1 \leq k \leq n$, where $d = 2/n$ and $u_k \approx u(x_k, t)$. Due to the periodic boundary condition, we need to set $u_0 = u_n$, $u_{-1} = u_{n-1}$, $u_{n+1} = u_1$ and $u_{n+2} = u_2$ for $k = 1, 2, n-1$ and n . Solve this system of equations with the 3-step Adams-Bashforth method (together with a third order Runge-Kutta method for the first a few steps) using the time step $h = 1/(m\pi)$ for $m = 600$. Plot the initial condition and the solutions at $t = 1/\pi$ and $t = 3.6/\pi$ in one figure. Submit the programs and the figure.

2. Consider the method BDF3:

$$y_{j+1} - \frac{18}{11}y_j + \frac{9}{11}y_{j-1} - \frac{2}{11}y_{j-2} = \frac{6}{11}hf(t_{j+1}, y_{j+1}).$$

- Explain how the coefficient $6/11$ in the right hand side is obtained.
- Calculate the local truncation error and find the order of the method.
- Show that this method is zero-stable.
- If BDF3 is applied to $y' = ay$ (where $a < 0$ is a constant) with step size $h > 0$, the numerical solutions satisfy

$$y_j = C_1\lambda_1^j + C_2\lambda_2^j + C_3\lambda_3^j,$$

where λ_1, λ_2 and λ_3 are roots of a cubic polynomial. For $A(0)$ -stability, we need $|\lambda_k| < 1$ for $k = 1, 2, 3$. Use MATLAB to plot $|\lambda_k|$ as a function of $z = ah$ for $-5 < z < 0$.