

MA3514 — Assignment No. 2

1. For the following system

$$\begin{aligned}x' &= (1-t)x + y \\y' &= x + (1+t)y - \frac{1}{2} \\x(0) &= 1 \\y(0) &= 0,\end{aligned}$$

find the numerical solutions at $t_1 = h$ for $h = 0.2$, with (a) the implicit midpoint method, (b) the trapezoid method.

2. Consider the following linear system of ordinary differential equations

$$\begin{aligned}\frac{dy}{dt} &= iAy, \quad t > 0 \\y(0) &= y_0\end{aligned}$$

where y is a complex vector function of t (a column vector of length n), A is a real symmetric $n \times n$ matrix and $i = \sqrt{-1}$ is the imaginary number. For numerical computation, we use $t_j = jh$, where h is the step size and denote the numerical approximation of $y(t_j)$ by y_j .

- (a) Apply the trapezoid method to this system, write down the formula for y_{j+1} in terms of y_j .
- (b) Show that the numerical solutions obtained from the trapezoid method satisfy

$$y_{j+1}^H y_{j+1} = y_j^H y_j$$

where $a^H = \bar{a}^T$ is the transpose of the complex conjugate of a .

3. The Korteweg-de Vries (KdV) equation (1895)

$$u_t + uu_x + \delta^2 u_{xxx} = 0,$$

where δ is a constant, models water waves in a shallow canal. The KdV equation has soliton solutions which were first observed by J. Scott Russell on the Edinburgh-Glasgow canal in 1834. Many mathematical properties of the KdV equation were found after the initial numerical study of Zabusky and Kruskal in 1965. They solved the KdV equation for $\delta = 0.022$ and $0 < x \leq 2$ assuming periodic boundary condition: $u(x+2, t) = u(x, t)$ for all x , with the initial condition: $u(x, 0) = \cos(\pi x)$, and for $0 < t \leq 3.6/\pi$. We will repeat this calculation with our own method.

Discretizing x by $x_k = 2k/n$ for $n = 200$ and $1 \leq k \leq n$, the KdV equation is approximated by the following system of ODEs:

$$\frac{du_k}{dt} + u_k \frac{u_{k+1} - u_{k-1}}{2d} + \frac{\delta^2}{2d^3} (u_{k+2} - 2u_{k+1} + 2u_{k-1} - u_{k-2}) = 0,$$

for $1 \leq k \leq n$, where $d = 2/n$ and $u_k \approx u(x_k, t)$. Due to the periodic boundary condition, we need to set $u_0 = u_n$, $u_{-1} = u_{n-1}$, $u_{n+1} = u_1$ and $u_{n+2} = u_2$ for $k = 1, 2, n-1$ and n . Solve this system of equations with the 3-step Adams-Bashforth method (together with a third order Runge-Kutta method for the first a few steps) using the time step $h = 1/(m\pi)$ for $m = 600$. Plot the initial condition and the solutions at $t = 1/\pi$ and $t = 3.6/\pi$ in one figure. Submit the programs and the figure.

4. Determine the local truncation error of BDF3. Therefore, show that it is a third order method.
5. For the following multi-step methods, determine the local truncation error (thus, the order of the method) and analyze the zero stability.

(a) $y_{j+1} - 2y_j + \frac{5}{4}y_{j-1} - \frac{1}{4}y_{j-2} = \frac{h}{4}f_{j-2}$.

(b) $y_{j+1} - 3y_j + 2y_{j-1} = h(f_{j+1} + f_j)$

6. To show that a numerical method is A(0)-stable, we apply the method to $y' = ay$ (where $a < 0$ is a constant) for step size $h > 0$, and show that $y_n \rightarrow 0$ as $n \rightarrow \infty$. For BDF2, the solution y_n satisfies

$$y_n = A\lambda_1^n + B\lambda_2^n,$$

where λ_1, λ_2 are the roots of a quadratic polynomial, A and B are constants. To be A(0)-stable, we need $|\lambda_1| < 1$ and $|\lambda_2| < 1$ for any $ah < 0$. Find λ_1 and λ_2 as functions of ah , and use MATLAB to plot $|\lambda_1|$ and $|\lambda_2|$ for $-5 < ah < 0$.