

## MA3514 — Assignment No. 1

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1. Apply the midpoint method to the following initial value problem for  $v = v(t)$  and  $w = w(t)$ :

$$\begin{aligned}v'' &= v + wv', & t > 0 \\w' &= -w + v^2 + t, & t > 0 \\v(0) &= 1, & v'(0) = 0, & w(0) = -1,\end{aligned}$$

with the step size  $h = 0.2$ , find the approximate values of  $v(0.2)$ ,  $v'(0.2)$  and  $w(0.2)$ .

2. Apply the general explicit second order Runge-Kutta method to the following system

$$\begin{cases} x' = -y, & x(0) = 1 \\ y' = x, & y(0) = 0 \end{cases}$$

with step size  $h$ . If we denote the numerical solutions at  $t_n = nh$  by  $x_n$  and  $y_n$ , find  $\lim_{n \rightarrow \infty} (x_n^2 + y_n^2)$ .

3. For the differential equation  $y' = f(t, y)$ , the local truncation error of the general 2nd order Runge-Kutta method (with parameter  $\alpha$ ) is

$$T_{j+1} = \frac{h^3}{4} \left[ \frac{2-3\alpha}{3} y'''(t_j) + \alpha y''(t_j) \frac{\partial f(t_j, y(t_j))}{\partial y} \right] + O(h^4).$$

Verify this result for the special case of  $\alpha = 1$ .

4. The following is another third order Runge-Kutta method for  $y' = f(t, y)$ :

$$\begin{aligned}k_1 &= f(t_j, y_j) \\k_2 &= f\left(t_j + \frac{h}{2}, y_j + \frac{h}{2}k_1\right) \\k_3 &= f\left(t_j + \frac{3}{4}h, y_j + \frac{3}{4}hk_2\right) \\y_{j+1} &= y_j + \frac{h}{9}(2k_1 + 3k_2 + 4k_3).\end{aligned}$$

Describe an embedded Runge-Kutta method based on the above third order method and a related second order Runge-Kutta method. Use this method to calculate the first time step  $t_1$  and the numerical solution  $y_1$  for

$$\begin{aligned}y' &= y - ty^2, & t > 0 \\y(0) &= 1,\end{aligned}$$

based on the initial step size  $h = 0.1$  and the error tolerance  $\epsilon = 10^{-3}$ .

5. For the following system

$$\begin{aligned}x' &= (1-t)x + y \\y' &= x + (1+t)y - \frac{1}{2} \\x(0) &= 1 \\y(0) &= 0,\end{aligned}$$

find the numerical solutions at  $t_1 = h$  for  $h = 0.2$ , with (a) the implicit midpoint method, (b) the trapezoid method.

6. Consider the following linear system of ordinary differential equations

$$\begin{aligned}\frac{dy}{dt} &= iAy, \quad t > 0 \\y(0) &= y_0\end{aligned}$$

where  $y$  is a complex vector function of  $t$  (a column vector of length  $n$ ),  $A$  is a real symmetric  $n \times n$  matrix and  $i = \sqrt{-1}$  is the imaginary number. For numerical computation, we use  $t_j = jh$ , where  $h$  is the step size and denote the numerical approximation of  $y(t_j)$  by  $y_j$ .

- (a) Apply the trapezoid method to this system, write down the formula for  $y_{j+1}$  in terms of  $y_j$ .
- (b) Show that the numerical solutions obtained from the trapezoid method satisfy

$$y_{j+1}^H y_{j+1} = y_j^H y_j$$

where  $a^H = \bar{a}^T$  is the transpose of the complex conjugate of  $a$ .