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MA 3513 Exam Solutions

1. The standard floating point numbers are

$$\pm (1.m_1 m_2 \cdots m_{23})_2 \times 2^q$$

In the interval $[2^q, 2^{q+1})$, the gap between two nearby floating point numbers is $\frac{1}{2^{23}} \times 2^q = 2^{q-23}$.

If $q = 23$, then the gap is 1.

Thus, if $x \in [2^{23}, 2^{24})$,

$$y = fl(x + 0.6) = x + 1$$

and $z = 1$.

2. $x_0 = -1, \quad f(x_0) = -1.6321$

$x_1 = -0.5, \quad f(x_1) = -0.3935$

$$x_2 = x_1 - \frac{f(x_1) \cdot (x_1 - x_0)}{f(x_1) - f(x_0)} = -0.3412$$

3. For $f(x) = 1$, $\quad c_1 + c_2 = \int_0^1 x^{-\frac{1}{3}} dx = \frac{3}{2}$

$f(x) = x$, $\quad c_1 x_1 + c_2 x_2 = \int_0^1 x^{\frac{2}{3}} dx = \frac{3}{5}$

$f(x) = x^2$, $\quad c_1 x_1^2 + c_2 x_2^2 = \int_0^1 x^{\frac{5}{3}} dx = \frac{3}{8}$

$f(x) = x^3$, $\quad C_1 x_1^3 + C_2 x_2^3 = \int_0^1 x^{\frac{8}{3}} dx = \frac{3}{11}$

Assume $\quad x_j^2 + \alpha x_j + \beta = 0 \quad$ for $j = 1, 2$,

then:

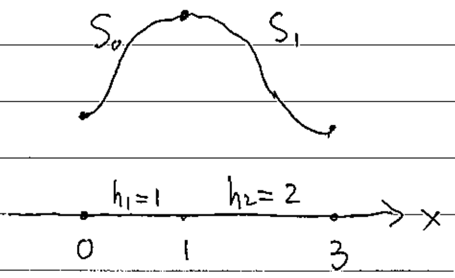
$$\begin{cases} \frac{3}{8} + \frac{3}{5}\alpha + \frac{3}{2}\beta = 0 \\ \frac{3}{11} + \frac{3}{8}\alpha + \frac{3}{5}\beta = 0 \end{cases}$$

$\Rightarrow \quad \alpha = -0.9091, \quad \beta = 0.1136$

Solve $\quad x^2 + \alpha x + \beta = 0 \quad$ gives

$x_1 = 0.1496, \quad x_2 = 0.7595$

4.



We only have to solve

$y_0'' = S''(x_0)$

$y_1'' = S''(x_1)$

Equation from the continuity of $S'(x_0) = S'(x_2)$

$$h_0 y_{-1}'' + 2(h_0 + h_1) y_0'' + h_1 y_1'' = 6\left[\frac{y_1 - y_0}{h_1} - \frac{y_0 - y_{-1}}{h_0}\right]$$

Due to the periodicity. $\quad h_0 = h_2 = 2$

$y_{-1}'' = y_1'', \quad y_{-1} = y_1$

$\Rightarrow: \quad 6 y_0'' + 3 y_1'' = 9$

Equation at x_1:

$$h_1 y_0'' + 2(h_1+h_2) y_1'' + h_2 y_2'' = 6\left[\frac{y_2-y_1}{h_2} - \frac{y_1-y_0}{h_1}\right]$$

Here: $y_2 = y_0$, $y_2'' = y_0''$

$\Rightarrow$: $3y_0'' + 6y_1'' = -9$

thus: $\begin{cases} 2y_0'' + y_1'' = 3 \\ y_0'' + 2y_1'' = -3 \end{cases}$

$\Rightarrow$: $y_0'' = 3$, $y_1'' = -3$

thus: $S''(0) = S''(x_0) = y_0'' = 3$.

5. Let $T(N)$ be the number of operations to calculate the N numbers f_j : $0 \le j < N$.

Thus, $T(1) = 1$.

$$T(N) = 2T\left(\frac{N}{2}\right) + 4 \times \frac{N}{2} = 2T\left(\frac{N}{2}\right) + 2N.$$

Here 4 operations are needed for: $a_j \alpha_j$, $b_j \beta_j$ and "+" and "−".

Thus,
$$\begin{aligned} T(N) &= 2T\left(\frac{N}{2}\right) + 2N \\ &= 2\left[2T\left(\frac{N}{2^2}\right) + 2\cdot\frac{N}{2}\right] + 2N, \\ &= 2^2 T\left(\frac{N}{2^2}\right) + 2\times 2N \\ &= 2^2\left[2T\left(\frac{N}{2^3}\right) + 2\cdot\frac{N}{2^2}\right] + 2\times 2N \end{aligned}$$

$= 2^3 T\left(\frac{N}{2^3}\right) + 3 \times 2N = \cdots$

$= 2^p T\left(\frac{N}{2^p}\right) + p \times 2N$

$= N T(1) + 2N \log_2 N = 2N \log_2 N + N$

since

$T(1) = 1$.

6. We have $PA = LU$, where

$$L = \begin{bmatrix} 1 & & \\ 2/3 & 1 & \\ 1/3 & -1/7 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 3 & 2 & 1 \\ & -\frac{2}{3} & \frac{4}{3} \\ & & -\frac{8}{7} \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

7. (a). $A = QR$, there are more than one solutions. One solution is

$$Q = \frac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \quad R = \begin{bmatrix} -3 & 3 \\ 0 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R_1 \\ \hline 0 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} -3 & 3 \\ 0 & 6 \end{bmatrix}$$

(b). $A^T x = b = R^T Q^T x = b$

$R^T y = b$ for $y = Q^T x = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\Rightarrow R_1^T \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = b$ $\quad y_3$ arbitrary

$\Rightarrow y_1 = -1, \quad y_2 = \frac{2}{3}$

For $\min\|x\| = \min\|y\|$, we choose $y_3 = 0$

$$\Rightarrow \quad y = \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2/3 \\ 0 \end{bmatrix} \Rightarrow x = Qy = \begin{bmatrix} -\frac{1}{9} \\ \frac{10}{9} \\ -4/9 \end{bmatrix}.$$

8 (a) $s = \dfrac{x_0^T A x_0}{x_0^T x_0} = 1$

Solve w from: $(A - sI)w = x_0$

$$\Rightarrow \quad w = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow x_1 = \frac{w}{\|w\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

(b) $B = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$, two eigenvalues of B are

$\lambda_1 = 1.3820, \quad \lambda_2 = 3.6180$

$\Rightarrow$ Wilkinson's shift $= 1.3820$

(c). $A = LU$, $L = \begin{bmatrix} 1 & & \\ * & 1 & \\ * & * & 1 \end{bmatrix}$ is invertible

$\tilde{A} = L^{-1}(LU)L = L^{-1}AL$

if $Ax = \lambda x \Rightarrow AL(L^{-1}x) = \lambda x$

$\Rightarrow L^{-1}AL(L^{-1}x) = \lambda(L^{-1}x)$

$\Rightarrow \tilde{A}y = \lambda y, \quad y = L^{-1}x$

$\Rightarrow A, \tilde{A}$ both have eigenvalue λ.