City University of Hong Kong

Course code & title:	MA3513 Elementary Numerical Methods
Session:	Semester A, 2008-2009
Time allowed:	Three hours

This paper has FOUR pages. (Including this page)

Instructions to candidates:

- Attempt all **EIGHT** questions.
- Start each question on a new page.
- Show all working.

Materials, aids & instruments permitted to be used during examination:

- Non-programmable portable battery operated calculator.

1. (8 marks) Let x, y and z be single precision floating point numbers, and assume that y and z are calculated by $y = x + 0.6$ and $z = y - x$, using floating point arithmetics. Give an example to show that we can get $z = 1$.

2. (8 marks) Consider the function $f(x) = e^x + 2x$. If the secant method is used to solve $f(x) = 0$ with $x_0 = -1$, $x_1 = -0.5$, find the next iteration x_2.

3. (8 marks) It is known that the following quadrature formula

$$\int_0^1 \frac{1}{\sqrt[3]{x}} f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

has no error when f is replaced by any polynomial of x with a degree ≤ 3. Find x_1 and x_2.

4. (8 marks) Given $n+1$ points (x_j, y_j) for $0 \leq j \leq n$, if $y_0 = y_n$, then we can find a cubic spline function $S(x)$ satisfying $S(x_j) = y_j$ (for all j) and the periodic boundary condition:

$$S'(x_0) = S'(x_n), \quad S''(x_0) = S''(x_n).$$

If $S(x)$ is the cubic spline function with periodic boundary condition for the three points given below, find $S''(0)$. Some useful results on cubic splines are listed in

x_j	0	1	3
y_j	1	2	1

the appendix.

5. (8 marks) Consider a recursive method to calculate N numbers: f_j for $0 \leq j < N$, where $N = 2^p$ and p is a non-negative integer.

   - If $p = 0$ (i.e. $N = 1$), we calculate f_0 using one operation.
   - If $p > 0$, we first calculate α_j and β_j (for $j = 0$, 1, ..., $\frac{N}{2} - 1$) using the same method for computing $\{f_j\}$, and then evaluate $\{f_j\}$ by

   $$f_j = a_j \alpha_j + b_j \beta_j, \quad f_{j+N/2} = a_j \alpha_j - b_j \beta_j \quad \text{for} \quad j = 0, 1, ..., \frac{N}{2} - 1,$$

   where a_j and b_j are given real numbers.

Find the total number of operations needed to calculate $\{f_j\}$.

6. (15 marks) Calculate the LU decomposition with partial pivoting for matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ 3 & 2 & 1 \end{bmatrix}.$$

Write down explicitly the matrices L, U and P.

7. Let matrix A and vector b be given by

$$A = \begin{bmatrix} 1 & -5 \\ 2 & 2 \\ -2 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

(a) (10 marks) Find a QR factorization of A.

(b) (10 marks) The linear system $A^T x = b$ (where A^T is the transpose of A, x is a vector of length 3) has infinitely many solutions. Find the solution with the minimum $||x||$.

8. Let matrix A and vector x_0 be given by

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(a) (10 marks) If x_0 is used as the initial vector in the Rayleigh quotient iteration method, find the next iteration x_1.

(b) (8 marks) If the QR method with Wilkinson's shift is used to find the eigenvalues of A, what is the Wilkinson's shift in the first iteration?

(c) (7 marks) If $A = LU$ is the LU decomposition of A and $\tilde{A} = UL$, show that the eigenvalues of A and $\tilde{A}$ are the same.

Appendix: cubic spline with natural boundaries

Let $S_j(x)$ be a cubic polynomial of x satisfying

$$S_j(x_{j-1}) = y_{j-1}, \quad S_j(x_j) = y_j, \quad S_j''(x_{j-1}) = y_{j-1}'', \quad S_j''(x_j) = y_j'',$$

then S_j can be written as

$$S_j(x) = Ay_{j-1} + By_j + \frac{h_j^2}{6}\left[(A^3 - A)y_{j-1}'' + (B^3 - B)y_j''\right]$$

where

$$\begin{aligned} h_j &= x_j - x_{j-1} \\ A &= \frac{x - x_j}{x_{j-1} - x_j} = -\frac{x - x_j}{h_j} \\ B &= \frac{x - x_{j-1}}{x_j - x_{j-1}} = \frac{x - x_{j-1}}{h_j} = 1 - A. \end{aligned}$$

Furthermore, the condition $S_j'(x_j) = S_{j+1}'(x_j)$ gives

$$h_j y_{j-1}'' + 2(h_j + h_{j+1})y_j'' + h_{j+1}y_{j+1}'' = 6\left[\frac{y_{j+1} - y_j}{h_{j+1}} - \frac{y_j - y_{j-1}}{h_j}\right].$$

— End —