City University of Hong Kong

Course code & title:	MA3513 Elementary Numerical Methods
Session:	Semester A, 2006-2007
Time allowed:	Three hours

This paper has THREE pages. (Including this page)

Instructions to candidates:

- Attempt all **SEVEN** questions.
- Start each question on a new page.
- Show all working.

Materials, aids & instruments permitted to be used during examination:

- Non-programmable portable battery operated calculator.

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1. (15 marks) To solve a nonlinear equation $f(x) = 0$ numerically, we may use the values of f, f' and f''. For a given initial guess x_0, we obtain x_1 by solving $P(x) = 0$, where $P(x)$ is the first three terms of the Taylor series:

$$P(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2.$$

   Since the quadratic equation $P(x) = 0$ may have two solutions, x_1 is chosen to be one which is closer to x_0.

   For $f(x) = x + 2\ln(x)$ and $x_0 = 1$, find x_1 by the above method.

2. Let $f(x) = \ln(x)/x$.

   (a) (7 marks) Find an approximate value of $f(1.1)$ based on the polynomial interpolation of f at $x = 1$, $x = 1.2$ and $x = 1.4$.

   (b) (8 marks) Find an approximation of

$$\int_1^2 f(x)dx$$

   based on the 3-point Gauss-Legendre quadrature formula.

3. (15 marks) For a general $n \times n$ matrix A, the algorithm for computing LU decomposition with partial pivoting (i.e., $PA = LU$) follows the steps as $k = 1, 2, ..., n-1$ and outputs L and U in the memory space of A.

   If for a 3×3 matrix A, the algorithm exchanges row 1 with row 2 in the first step ($k = 1$) and produces the following matrix

$$\begin{bmatrix} 1 & 5 & 6 \\ \frac{1}{2} & 1 & 7 \\ \frac{1}{3} & 3 & 6 \end{bmatrix}$$

   after the first step is completed. Carry out the second step of the algorithm and find the matrices L, U and P.

4. (15 marks) Let A be the following 3×3 matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ 3 & 2 & 1 \end{bmatrix}.$$

   Find a matrix T with a zero $(3, 1)$ entry, such that A and T have the same eigenvalues.

5. Let matrix A and vector b be given by

$$A = \begin{bmatrix} 2 & 0.5 \\ -1 & 1 \\ 2 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 1 \\ 8 \end{bmatrix}.$$

   (a) (8 marks) Find a QR factorization of A.

   (b) (7 marks) Solve the least squares problem

$$\min_x ||Ax - b||.$$

   You need to calculate both the minimum value of $||Ax - b||$ and the value of x at which the minimum is obtained.

6. For two sets of real coefficients $\{c_k\}$ and $\{d_k\}$, $0 \le k < N$, where N is a positive integer, we can calculate their discrete Fourier transforms using only one FFT for a complex vector of length N. Consider

$$F_j = \sum_{k=0}^{N-1} (c_k + id_k)e^{i2\pi jk/N}, \quad G_j = \sum_{k=0}^{N-1} (c_k - id_k)e^{i2\pi jk/N},$$

   for $j = 0,\ 1,\ \ldots,\ N-1$.

   (a) (8 marks) Show that $\overline{G}_0 = F_0$ and $\overline{G}_j = F_{N-j}$ for $j = 1, 2, \ldots, N-1$.

   (b) (7 marks) Let the discrete Fourier transforms of $\{c_k\}$ and $\{d_k\}$ be

$$f_j = \sum_{k=0}^{N-1} c_k e^{i2\pi jk/N}, \quad g_j = \sum_{k=0}^{N-1} d_k e^{i2\pi jk/N}, \quad 0 \le j < N.$$

   Express f_j and g_j in terms of $\{F_j\}$.

7. (10 marks) Let A be a singular 3×3 matrix. Assume that the first two columns of A are linearly independent. If $A = QR$ is the QR factorization of A, show that the last row of RQ is zero.

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