City University of Hong Kong

Course code & title:	MA3513 Elementary Numerical Methods
Session:	Semester A, 2005-2006
Time:	18:30 pm — 21:30 pm
Time allowed:	Three hours

This paper has THREE pages. (Including this page)

Instructions to candidates:

- Attempt all **SEVEN** questions.
- Start each question on a new page.
- Show all working.

Materials, aids & instruments permitted to be used during examination:

- Non-programmable portable battery operated calculator.

1. (15 marks) Find approximate values of the integral

$$I = \int_0^1 \frac{e^x}{1+x} dx$$

by (a) the elementrary Simpson rule; (b) the three-point Gauss-Legendre quadrature formula.

2. (15 marks) For the matrix A below, find the LU decomposition with partial pivoting, i.e. $PA = LU$.

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}.$$

3. (15 marks) Let matrix A and vector b be given by

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

(a) Find a QR factorization of A.

(b) Solve the least squares problem

$$\min_x ||Ax - b||.$$

You need to calculate both the minimum value of $||Ax - b||$ and the value of x at which the minimum is obtained.

4. (15 marks) For matrix A and the initial vector x_0 given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

find the next iteration x_1 by the method of Rayleigh quotient iteration.

5. (15 marks) To solve a nonlinear equation $f(x) = 0$ numerically, we may use three initial guesses: x_0, x_1 and x_2. To find the next iteration x_3, we first determine a quadratic polynomial $Q(y)$ that interpolates (y_0, x_0), (y_1, x_1) and (y_2, x_2) where $y_j = f(x_j)$ for $j = 0, 1, 2$. Notice that Q is a polynomial of y and $x = Q(y)$ is an approximation of the inverse function $x = f^{-1}(y)$. Once Q is obtained, we find x_3 as the intersection of $x = Q(y)$ with the x-axis. Namely, $x_3 = Q(0)$. This step can be repeated to calculate x_4 with x_1, x_2 and x_3, etc.

   For $f(x) = x^3 + 2x + 1$ and $x_0 = 1$, $x_1 = 0$, $x_2 = -1$, find the next iteration x_3.

6. (15 marks) For a set of coefficients $\hat{f}_0$, $\hat{f}_1$, ..., $\hat{f}_{N-1}$, where N is an even integer, we define

$$\begin{aligned} \alpha_j &= \hat{f}_0 + \hat{f}_2\omega^{2j} + \hat{f}_4\omega^{4j} + \dots + \hat{f}_{N-2}\omega^{(N-2)j}, \\ \beta_j &= \hat{f}_1 + \hat{f}_3\omega^{2j} + \hat{f}_5\omega^{4j} + \dots + \hat{f}_{N-1}\omega^{(N-2)j}, \end{aligned}$$

   for $j = 0, 1, 2, \dots, \frac{N}{2} - 1$ and $\omega = e^{i2\pi/N}$. For $m = -\frac{N}{2}, \dots, -2, -1$, we define

$$f_m = \sum_{k=0}^{N-1} \hat{f}_k \omega^{mk}.$$

   Express f_m in terms of α_j, β_j and ω. Give a detailed derivation for your result.

7. (10 marks) For a general 3×3 matrix A, show that we can find three Householder reflections H_1, H_2 and H_3, such that

$$H_3 H_1 A H_2 = B = \begin{bmatrix} * & * & 0 \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}.$$

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