City University of Hong Kong

Course code & title:	MA3513 Numerical Methods I
Session:	Semester A, 2003-2004
Date:	December 9, 2003
Time:	18:30 pm — 21:30 pm
Time allowed:	Three hours

This paper has THREE pages. (Including this page)

Instructions to candidates:

- Attempt all **EIGHT** questions.
- Start each question on a new page.
- Show all working.

Materials, aids & instruments permitted to be used during examination:

- Non-programmable portable battery operated calculator.

1. Assume that in a special floating point number system, an arbitrary non-zero floating point number x can be written as
$$x = \pm \left(1 + \frac{m_1}{2} + \frac{m_2}{4} + \frac{m_3}{8} + \frac{m_4}{16}\right) \times 2^q,$$
where m_j is either 1 or 0, and q is an integer.

   (a) (5 marks) What is the distance between two nearby floating point numbers in the interval (8,16)?

   (b) (5 marks) For any non-zero real number y, if $fl(y)$ is the floating point number nearest to y, we have
$$\left|\frac{y - fl(y)}{y}\right| \leq \epsilon_M.$$
   What is the smallest value of ϵ_M?

2. (12 marks) A three-point numerical method for solving $f(x) = 0$ provides a formula for x_{k+3} assuming x_k, x_{k+1} and x_{k+2} are given. One such method is the Linear Fractional method. It constructs a rational approximation of $f(x)$ and uses its zero as the next iteration. More precisely and for $k = 0$, we start with x_0, x_1 and x_2 as given, and finds
$$Q(x) = \frac{x - a}{bx - c}$$
for some coefficients a, b and c, such that
$$Q(x_j) = f(x_j) \quad \text{for} \quad j = 0, 1, 2,$$
then calculates x_3 from $Q(x_3) = 0$. Use this method to find x_3 for $f(x) = x^3 - 3x + 1$, assuming $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$.

3. Define the points (x_j, y_j) for $j = 0, 1, 2, ...$, by
$$x_j = j, \quad y_j = 1 - j^3.$$

   (a) (5 marks) If $P_2(x)$ interpolates the first three points, i.e., (x_j, y_j) for $j = 0, 1, 2$, find $P_2(-1)$.

   (b) (5 marks) If $P_{100}(x)$ interpolates the first 101 points, i.e., (x_j, y_j) for $j = 0, 1, ..., 100$, find $P_{100}(-1)$.

4. (10 marks) The following integration formula
$$\int_0^1 x^{-1/3} f(x)\, dx \approx c_0 f(x_0) + c_1 f(x_1)$$
is designed such that it has no error when f is a polynomial of degree ≤ 3. It turns out that x_0, x_1 are zeros of a quadratic polynomial $P(x) = x^2 + ax + b$, such that
$$\int_0^1 x^{-1/3} P(x) Q(x)\, dx = 0,$$
where Q is any polynomial of degree ≤ 1. Find $P(x)$.

5. (13 marks) For a given even integer N and some given coefficients c_k (for $-N/2 \leq k < N/2$), we define a function f by

$$f(x) = \sum_{k=-N/2}^{N/2-1} c_k e^{ikx}.$$

Describe how the Fast Fourier Transform can be used to evaluate the derivative $f'(x_j)$ for $j = 0, 1, ..., N-1$, where

$$x_j = \frac{2\pi j}{N}.$$

6. (15 marks) Find the LU decomposition with partial pivoting for

$$A = \begin{bmatrix} 3 & 5 & 7 \\ -3 & 0 & 15 \\ 9 & 6 & 3 \end{bmatrix}.$$

7. (15 marks) Find a Householder reflection H, such that

$$H \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 3 \\ 2 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & * \\ 0 & * \\ * & * \\ 0 & * \\ 1 & * \end{bmatrix}.$$

Calculate the entries denoted as $*$.

8. (15 marks) Let A be a 3×3 matrix and a_{ij} be its (i, j) entry. Assume that $a_{31} = 0$ and $a_{21} \neq 0$. Let $A = QR$ be the QR factorization of A and define $\hat{A} = RQ$. Show that the $(3, 1)$ entry of $\hat{A}$ is also 0.

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