

1. A real symmetric matrix  $A$  is positive definite, if for any non-zero column vector  $w$ , we have  $w^T A w > 0$ . Show that if  $A$  is symmetric positive definite matrix, then  $A$  has a Cholesky decomposition. That is  $A = SS^T$ , where  $S$  is a lower triangular matrix with positive diagonal entries. To prove this, you need the following steps:

- (a) Re-write  $A$  as

$$A = \begin{bmatrix} a & b^T \\ b & C \end{bmatrix}$$

where  $a$  is the  $(1, 1)$  entry of  $A$ ,  $b$  is an  $(n-1) \times 1$  column vector,  $b^T$  is the transpose of  $b$  and  $C$  is an  $(n-1) \times (n-1)$  matrix. Find  $x > 0$ ,  $y$  and  $Z$ , such that

$$A = \begin{bmatrix} x & \\ y & I \end{bmatrix} \begin{bmatrix} 1 & \\ & Z \end{bmatrix} \begin{bmatrix} x & y^T \\ & I \end{bmatrix}$$

where  $y$  is a column vector of length  $n-1$  and  $Z$  is an  $(n-1) \times (n-1)$  matrix.

- (b) Show that  $Z$  is also symmetric positive definite.  
 (c) As in an induction proof, assume  $Z$  has a Cholesky decomposition, that is  $Z = S_1 S_1^T$ , then find  $S$  such that  $SS^T = A$ .

2. Find a Cholesky decomposition of

$$A = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 11 \end{bmatrix}.$$

3. Find the values of  $\beta$ , such that the matrix  $A$  below is positive definite.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 13 & 23 \\ 4 & 23 & \beta \end{bmatrix}.$$

4. Find the LU decomposition with partial pivoting (i.e. PA=LU) for the matrices below.

$$(a). \quad A = \begin{bmatrix} 1 & 1 & 7 \\ 0 & 1 & 3 \\ -2 & 2 & 4 \end{bmatrix}, \quad (b). \quad A = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 2 & 4 \\ 0 & 6 & 3 \end{bmatrix}, \quad (c). \quad A = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 1 & 0 \\ 1 & -1 & 1/2 \end{bmatrix}.$$

5. Let  $A$  be a tridiagonal matrix

$$A = \begin{bmatrix} a_1 & c_1 & & & \\ b_1 & a_2 & c_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ & & & b_{n-1} & a_n \end{bmatrix}.$$

- (a) Write a MATLAB function for its LU decomposition (without partial pivoting) using only three vectors. For input, you have the three vectors  $a$ ,  $b$  and  $c$  that define the matrix  $A$ . For output, the entries of  $L$  and  $U$  are stored in  $b$  and  $a$ ,  $c$ , respectively. The original values of  $a$ ,  $b$  and  $c$  are destroyed. That is

$$L = \begin{bmatrix} 1 & & & & & & \\ b_1 & 1 & & & & & \\ & b_2 & \ddots & & & & \\ & & \ddots & \ddots & & & \\ & & & b_{n-1} & 1 & & \\ & & & & & & 1 \end{bmatrix}, \quad U = \begin{bmatrix} a_1 & c_1 & & & & & \\ & a_2 & c_2 & & & & \\ & & \ddots & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & \ddots & c_{n-1} & \\ & & & & & & a_n \end{bmatrix}.$$

- (b) Write a MATLAB function for solving  $Ly = f$  using only two vectors. One vector for the matrix  $L$  and the other vector  $f$ . The solution  $y$  is written in  $f$ .
- (c) Write a MATLAB function for solving  $Ux = y$  using only three vectors. Two vectors for the matrix  $U$  and the vector  $y$ . The solution  $x$  is written in  $y$ .
- (d) Write a main program in MATLAB that solves  $Ax = f$  for a tridiagonal matrix based on its LU decomposition. Test you program on the following  $7 \times 7$  system:

$$\begin{bmatrix} 1 & 1 & & & & & \\ 1 & 0 & 2 & & & & \\ & 1 & 0 & \ddots & & & \\ & & \ddots & \ddots & 6 & & \\ & & & & 1 & 0 & \end{bmatrix} x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$