

MA3513 — Assignment # 5

1. For a given sequence c_0, c_1, \dots, c_n , the Discrete Cosine Transform produces the new sequence f_0, f_1, \dots, f_n by

$$f_j = \sum_{k=0}^n c_k \cos\left(\frac{jk\pi}{n}\right) \quad \text{for } j = 0, 1, \dots, n.$$

How do you use FFT to calculate f_0, f_1, \dots, f_n ?

2. Write down the matrix T in Problem 1 that transforms c_j (for $j = 0, 1, \dots, n$) to f_j (for $j = 0, 1, \dots, n$), that is

$$\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix} = T \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Use MATLAB, find the relationship between

$$T \quad \text{and} \quad \frac{n}{2}T^{-1}.$$

3. For $i = \sqrt{-1}$ and 8 real numbers c_0, c_1, \dots, c_7 , we define

$$g_j = \sum_{k=0}^3 [c_{2k} + ic_{2k+1}] e^{ijk2\pi/4}, \quad h_j = \sum_{k=0}^3 [c_{2k} - ic_{2k+1}] e^{ijk2\pi/4} \quad \text{for } j = 0, 1, 2, 3.$$

- (a) Show that h_j is the complex conjugate of g_{4-j} , i.e.,

$$h_j = \bar{g}_{4-j} \quad \text{for } j = 0, 1, 2, 3.$$

- (b) Let f_0, f_1, \dots, f_7 be given by

$$f_j = \sum_{k=0}^7 c_k e^{ijk2\pi/8} \quad \text{for } j = 0, 1, \dots, 7,$$

find the formulas for $\{f_j\}_{j=0}^7$ in terms of g_0, g_1, g_2, g_3 .

4. Let $\{f_k\}$ be given by the discrete Fourier transform:

$$f_k = \sum_{j=0}^{N-1} c_j e^{\frac{i2\pi jk}{N}} \quad \text{for } k = 0, 1, \dots, N-1,$$

where $\mathbf{i} = \sqrt{-1}$ and $N = 8$. It is known that

$$\begin{aligned} a_0 &= 1, & a_1 &= 2, & a_2 &= 3, & a_3 &= 4, \\ b_0 &= 5, & b_1 &= 6, & b_2 &= 7, & b_3 &= 8, \end{aligned}$$

where $\{a_k\}$ and $\{b_k\}$ (for $k = 0, 1, 2, 3$) are defined as follows:

$$\begin{aligned} a_k &= c_0 + c_2 e^{\frac{i2\pi k}{4}} + c_4 e^{\frac{i2\pi 2k}{4}} + c_6 e^{\frac{i2\pi 3k}{4}} \\ b_k &= c_1 + c_3 e^{\frac{i2\pi k}{4}} + c_5 e^{\frac{i2\pi 2k}{4}} + c_7 e^{\frac{i2\pi 3k}{4}}. \end{aligned}$$

Find f_0, f_1, \dots, f_7 .

5. If for some vector $c = [c_0, c_1, \dots, c_7]$, its discrete Fourier transform is

$$\begin{array}{c|c|c|c|c|c|c|c} f_0 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 \\ \hline 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \end{array}$$

define and find the discrete Fourier transforms of $[c_0, c_2, c_4, c_6]$ and $[c_1, c_3, c_5, c_7]$.

6. The function

$$f(x) = \frac{1}{1 + 0.5 \cos(x)}$$

is 2π periodic. We can discretize x and f by

$$x_j = \frac{j2\pi}{N}, \quad f_j = f(x_j), \quad j = 0, 1, 2, \dots, N-1.$$

Write a MATLAB program (based on `fft` and `ifft`) to calculate its approximate second order derivative

$$f''(x_j), \quad j = 0, 1, 2, \dots, N-1.$$

Also, find the f'' analytically and calculate the maximum error (in MATLAB)

$$E_N = \max_{0 \leq j < N} |f''(x_j)^{(analytic)} - f''(x_j)^{(numerical)}|$$

for $N = 4, 8, 16, 32$. Submit the MATLAB program and the results of E_N .