

1. Find approximate values of

$$I = \int_1^2 \frac{\cos(x)}{1+x} dx$$

based on

- the composite trapezoidal rule with grid size  $h = 1/6$ ;
  - the composite Simpson's method with a grid size of  $h = 1/6$ ;
  - the three point Gauss-Legendre formula.
2. Let  $\phi(h)$  be the approximation to  $L = \int_a^b f(x)dx$  by the Trapezoidal rule using the points:  $x_0 = a$ ,  $x_j = a + jh$ ,  $x_n = b$  for  $h = (b - a)/n$ . This method has a second order of accuracy:

$$\phi(h) - L = C_2 h^2 + \dots$$

where  $C_2$  is a constant. If for  $n = 100$  and  $n = 300$ , the Trapezoidal rule gives the values 4 and 4.8, respectively. Find a more accurate approximation for  $L$ .

3. If the composite Simpson's method is used to approximate

$$I = \int_2^3 x^4 dx$$

with  $h = 1/100$ , what is the absolute error? If the composite trapezoidal rule is used instead, what are the lower and upper bounds of the absolute error?

4. Similar to Simpson's rule based on quadratic polynomial interpolation, we can use polynomial interpolations of higher degree for numerical integration. For  $x_j = x_0 + jh$ , derive the following formula (called Simpson's  $\frac{3}{8}$  rule)

$$\int_{x_0}^{x_3} f(x)dx \approx \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$$

based on  $f(x) \approx P_3(x)$ , where  $P_3$  interpolates  $f$  at  $x_0, x_1, x_2$  and  $x_3$ .

5. It is sometimes useful to estimate the integral of a function on the interval  $(x_0, x_1)$  from the values of the function outside the interval. This is called "extrapolation". For  $x_j = x_0 + jh$ , derive the following formula

$$\int_{x_0}^{x_1} f(x)dx \approx \frac{h}{12}[23f(x_1) - 16f(x_2) + 5f(x_3)],$$

based on  $f(x) \approx P_2(x)$ , where  $P_2$  interpolates  $f$  at  $x_1, x_2$  and  $x_3$ .

6. Consider the following numerical integration formula

$$\int_0^1 \sqrt{x}f(x) dx \approx c_0f(x_0) + c_1f(x_1).$$

If the above formula has no error when  $f$  is any polynomial of degree  $\leq 3$ , find  $c_0, c_1, x_0$  and  $x_1$ .

7. How do you choose  $c_0$ ,  $c_1$  and  $x_1$  such that the formula

$$\int_0^{\infty} f(x)e^{-x^2} dx \approx c_0f(0) + c_1f(x_1)$$

is as accurate as possible? The following formulas may be useful.

$$2 \int_0^{\infty} x^2 e^{-x^2} dx = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}, \quad \int_0^{\infty} x e^{-x^2} dx = \frac{1}{2}$$

8. Design an integration formula of the form:

$$\int_{-1}^1 f(x)dx \approx c_0f(-1) + c_1f(x_1) + c_2f(x_2) + c_3f(1),$$

such that the above formula has no error for polynomials with a degree as high as possible. You may assume certain symmetry.

9. Write a MATLAB program to calculate the zeros of the Legendre polynomial  $L_n(x)$  for a given integer  $n \geq 2$  by Newton's method. You need the recurrence formula to evaluate  $L_n(x)$  and the following formula for  $L'_n(x)$

$$L'_n(x) = \frac{n}{x^2 - 1} [xL_n(x) - L_{n-1}(x)].$$

For the  $j$ -th zero of  $L_n(x)$ , you can use the following initial guess:

$$x_j^{(0)} = \cos \frac{(j - 0.25)\pi}{n + 0.5}.$$

Submit your program and the results for  $n = 15$  and  $n = 16$ .