

1. Let f be a function of x and we know that

$$f(0) = 1, f(0.1) = 1.2, f(0.2) = 0.9.$$

Find an approximate value of $f(0.15)$ based on a polynomial interpolation of f at the three given points.

2. Let $y(x)$ be a function of x and $h > 0$ be a given constant. Assume that the following values are known:

$$y_0 = y(0), y_1 = y(h), f_0 = y'(0), f_1 = y'(h).$$

Find an approximation of $y(2h)$, say $y_2 \approx y(2h)$, based on

$$y(2h) - y(h) = \int_h^{2h} y'(x) dx \approx \int_h^{2h} P_1(x) dx$$

where $P_1(x)$ is an approximation to $y'(x)$ based on a polynomial interpolating $(0, f_0)$ and (h, f_1) .

3. Let $P_{10}(x) = a_{10}x^{10} + a_9x^9 + \dots + a_1x + a_0$ be the polynomial interpolating the 11 points: $x_i = 0, 1, 2, \dots, 10$, such that

$$P_{10}(0) = 2, P_{10}(j) = 1, \quad \text{for } j = 1, 2, \dots, 10.$$

Find a_{10} and a_9 .

4. Let $\{x_0, x_1, \dots, x_n\}$ be $n + 1$ distinct numbers, where $n \geq 2$, find

$$\sum_{i=0}^n (2 - x_i^2) \prod_{\substack{j=0, \dots, n \\ j \neq i}} \frac{x - x_j}{x_i - x_j}$$

5. Let $S(x)$ be the natural cubic spline function that connects (x_j, y_j) in the table:

x_j	0	1	2	3
y_j	0	1	1	0.5

find $S(1.5)$.

6. Let $S(x)$ be the natural cubic spline function that connects (x_j, y_j) below:

x_j	-2	-1	1	2
y_j	0	1	2	1

find $S''(-1), S''(1)$.

7. (Clamped cubic spline) Given $n + 1$ points (x_j, y_j) for $0 \leq j \leq n$ (assuming $x_0 < x_1 < \dots < x_n$) and two constants y'_0 and y'_n , we can find a cubic spline function $S(x)$ satisfying

$$S'(x_0) = y'_0, \quad S'(x_n) = y'_n, \quad S(x_j) = y_j, \quad j = 0, 1, \dots, n,$$

where $S(x) = S_j(x)$ on (x_{j-1}, x_j) is a polynomial of degree at most three, and $S(x)$ has continuous second order derivative.

- (a) On the interval (x_{j-1}, x_j) , find $S_j(x)$ (a polynomial of degree at most 3) satisfying

$$S_j(x_{j-1}) = y_{j-1}, \quad S_j(x_j) = y_j, \quad S'_j(x_{j-1}) = y'_{j-1}, \quad S'_j(x_j) = y'_j.$$

- (b) Find a linear system of equations for $y'_j = S'(x_j)$, $j = 1, 2, \dots, n - 1$.
(c) Write a MATLAB program for clamped cubic spline (similar to the MATLAB program in section 3.2 of the lecture notes). For the five points (x_j, y_j) given by

$$(0, 3), (1, 1), (2, 1), (3, 2), (4, 0.5)$$

and the boundary conditions: $S'(0) = 0$ and $S'(4) = 0$, use the MATLAB program to plot $S(x)$. Submit your MATLAB program and the plot.