

MA3513 ELEMENTARY NUMERICAL METHODS, ASSIGNMENT NO. 2

1. If the bisection method is used to solve $f(x) = 0$ (the exact solution is x_*) starting from the interval $[a, b]$, where $a = -0.5$ and $b = 2.0$, a sequence $\{x_1, x_2, x_3, \dots\}$ is generated, where $x_1 = (b + a)/2$. For what values of n , the condition

$$|x_n - x_*| \leq 0.5 \times 10^{-8}$$

is definitely true.

2. Give a graphical demonstration that the equation

$$\tan(x) = x^2$$

has infinitely many roots. Determine one root approximately (up to 3 correct digits) by the bisection method.

3. (computer) Write a general bisection program in MATLAB and use this program to solve

$$x^3 - 2 \sin(x) = 0$$

on the interval $[0.5, 2]$.

4. Find the zero of $f(x) = e^{-x} - \cos(x)$ that is nearest to $\pi/2$ by Newton's method to at least three digits of accuracy. Use your own initial guess x_0 .
5. When the Newton's method is applied to some function f with the initial guess $x_0 = 1$, the value $x_1 = 0.5$ is obtained. It is also known that $f(1) = 2$. What is the slope of f at $x = 1$?

6. Consider the problem

$$x^{20} - 1 = 0.$$

$x_* = 1$ is a simple root of the above equation. Apply Newton's method with $x_0 = 0.5$, find x_1 , x_2 and x_3 .

7. If Newton's method is used to solve $f(x) = 0$ for $f(x) = 2 - 1/x^2$, with the initial guess $x_0 = 1$, find x_1 and x_2 . What is the limit of x_n as $n \rightarrow \infty$?
8. For Newton's method, find C_1 and C_2 by a Taylor expansion around x_* , such that

$$x_{k+1} - x_* = C_1(x_k - x_*)^2 + C_2(x_k - x_*)^3 + \dots$$

where x_* is the exact solution of $f(x) = 0$ and C_1, C_2 are functions of x_* only.

9. If x_* is a double zero of f , we have

$$0 = f(x_*) = f'(x_*) \neq f''(x_*).$$

We can also write $f(x) = (x - x_*)^2 g(x)$ for some function g . For Newton's method given as

$$x_{k+1} = \phi(x_k) = x_k - \frac{f(x_k)}{f'(x_k)},$$

show that $\phi'(x_*) \neq 0$. We also define the modified Newton's method as

$$x_{k+1} = \varphi(x_k) = x_k - 2 \frac{f(x_k)}{f'(x_k)}.$$

Show that $\varphi'(x_*) = 0$.

10. Use Newton's method on $f(x) = x^3 - x + 1$ with $x_0 = 1$, find x_1 . Use the secant method on $f(x)$ (with the same x_0 and x_1 from Newton's method), find x_2 .

11. Consider the following equation

$$x^2 = \cos(x).$$

(a) Find x_1 based on Newton's method and the initial guess $x_0 = 1$.

(b) Find x_2 based on the secant method and the two initial guesses $x_0 = 1$, $x_1 = 0.5$.

12. Calculate an approximate value for $\sqrt{8}$ using two steps of the secant method with $x_0 = 3$ and $x_1 = 2$ for solving $f(x) = x^2 - 8 = 0$.

13. (computer) Write a MATLAB program for the secant method. Besides the two initial guess x_0 and x_1 , the function should also input ϵ and δ , such that the function returns x_{k+1} when

$$\left| \frac{x_{k+1} - x_k}{x_{k+1}} \right| < \epsilon$$

or

$$|f(x_{k+1})| < \delta.$$

Use the program to find a zero of the function

$$f(x) = (x - 1)(x - 2)\dots(x - 20) - 10^{-8}x^{19}$$

in the interval $[20, 21]$.

14. Let $e_n > 0$ for $n = 0, 1, 2, \dots$ and $e_n \rightarrow 0$ as $n \rightarrow \infty$. If for some $C > 0$,

$$e_{n+1} = C e_n e_{n-1} e_{n-2} \quad \text{for } n = 2, 3, \dots$$

find $\alpha > 1$ and $\lambda > 0$, such that

$$e_{n+1} = \lambda e_n^\alpha.$$